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ON A PROBLEM CONNECTED WITH MATRICES OVER Z_3

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Abstract: In this note we give an explicit form of the matrix $A = (a_{ij})_{n\times n}$ with elements $a_{ij} \in Z_3$, which satisfy all conditions of some problem posed by Stewart M. Venit (see [3], p. 476 — Unsolved Problems). Moreover, we prove that if $\alpha_1, \alpha_2, \dots, \alpha_n$ are the characteristic roots of this matrix then for every prime number p the following congruence is true $\alpha_1^p + \alpha_2^p + \dots + \alpha_n^p \equiv 2n-1 \pmod p$.

1. Introduction

In [3] (p. 476 — Unsolved Problems — TYCMJ 186 — by Stewart M. Venit) one can find the following problem: For each positive integer n show that there is one and only one $n \times n$ matrix A satisfying the following conditions:

(C1) all entries of A are in the set $\{0, 1, 2\}$

(C2) the submatrix consisting of the first k rows and k columns of A has determinant equal to k for $k = 1, 2, \dots, n$.

(C3) all entries of A not on the main diagonal or not on the diagonals directly above or below are zero.

In the present note we prove that the matrix $A_n\,(a_{ij})_{n\times n}$, where $a_{ij} \in Z_3 = \{0, 1, 2\}$ and $a_{12} = a_{21} = 0$ given by

$$a_{ij} = a_{ji} = \begin{cases} 1, & \text{if } i = j = 1 \text{ or } |i-j| = 1 \text{ for } \max(i,j) \geq 3 \\ 2, & \text{if } i = j \geq 2 \\ 0, & \text{in the other cases and if } (i,j) = (1,2) \end{cases} \tag{1}$$

satisfies the conditions (C1)–(C3) and is determined uniquely.

2. Results

First, we prove the following

Theorem 1. *For each positive integer $n \geq 2$ there is exactly one of the matrix $A_n = (a_{ij})_{n\times n}$ with elements over Z_3 such that the conditions (C1)–(C3) are satisfied. The matrix A_n given by (1) has the following form:*

$$A_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}_{n\times n} \tag{2}$$

Proof of Theorem 1. It is easy to see that for $n = 2$ the matrix A_2 satifying the conditions (C1)–(C3) is the form

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and we see that the matrix A_2 is determined uniquely. For $n = 3$ we obtain that the matrix A_3 has the following form

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We note that the matrix A_3 is determined uniquely and the conditions (C1)–(C3) are satisfied. Further, we shall prove Theorem 1 by induction with respect to n. Suppose that $m \geq 3$ and the matrices A_n for $n \leq m$ has the form (2) and are determined uniquely. By inductive assumption it follows that the matrix A_{m+1} has the following form

$$A_{m+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 2 & y \\ 0 & 0 & \dots & 0 & x & z \end{pmatrix}_{(m+1)\times(m+1)} \tag{3}$$

where $x, y, z \in Z_3 = \{0, 1, 2\}$. Suppose that in (3) we have $x = y = 1$ and $z = 2$. Using Laplace's theorem to the first row of the matrix A_{m+1} we obtain

$$\det A_{m+1} = \det \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}_{m\times m}$$

On the other hand it is well-known (see [2], p. 39) that

$$\det A_{m+1} = m + 1. \tag{4}$$

By (4) and the inductive assumption it follows that the matrix A_{m+1} satisfies the conditions (C1)–(C3), if $x = y = 1$ and $z = 2$. Now, we can assume that the elements $x, y, z \in Z_3$ take different values than $x = y = 1$ and $z = 2$. Using Laplace's theorem to (3) with respect to the last row and by the inductive assumption we obtain

$$\det A_{m+1} = mz - xy\,(m-1)\,. \tag{5}$$

Consequently, we can consider the following equation generated by (5)

$$mz - xy\,(m-1) = m + 1 \tag{6}$$

where $x, y, z \in Z_3 = \{0, 1, 2\}$. Analyzing (6) we obtain, that this equation has exactly one solution in elements $x, y, z \in Z_3$, namely $x = y = 1$ and $z = 2$. Therefore the matrix A_{m+1} is determined uniquely. Hence the inductive proof is complete.

Now, we prove the following theorem:

Theorem 2. *Let A_n be the matrix defined by (1) and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the characteristic roots of A_n. Then for every prime number p, the following congruence*

$$\alpha_1^p + \alpha_2^p + \cdots + \alpha_n^p \equiv 2n - 1 \pmod{p} \tag{7}$$

holds.

Proof of Theorem 2. It is well-known that if $f \in Z\,[x]$ and $x_1, x_2, \ldots, x_n$ are the roots of f, then

$$S_{jp} \equiv S_j \pmod{p} \tag{8}$$

for $j = 1, 2 \ldots$ and every prime number p, where

$$S_k = x_1^k + x_2^k + \cdots + x_n^k.$$

The congruence (8) has been noticed without proof by E. Lucas in 1878. The proof of (8) one can find, for example in [1]. Substituting $j = 1$ in (8) and remarked that

$$S_1 = Tr A_n = 2n - 1$$

we obtain, that

$$S_p = \alpha_1^p + \alpha_2^p + \cdots + \alpha_n^p \equiv S_1 = 2n - 1 \pmod{p}$$

and the proof of the Theorem 2 is complete.

Substituting $n = p$ in (7), where p is a prime number we obtain the following

Corollary. *Let p be the a prime number and let α_j $j = 1, 2, \dots, p$ be the characteristic roots of the matrix A_p given by (1), then*

$$\alpha_1^p + \alpha_2^p + \cdots + \alpha_p^p \equiv -1 \pmod{p}.$$

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References

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