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SOME REMARKS ON HERON TRIANGLES

Alpar-Vajk Kramer (Mediaş), Florian Luca (Bielefeld)

Abstract. In this note, we collect a few facts about Heron triangles. For example, we show that there exist infinitely many pairs of incongruent Heron triangles having the same area and semiperimeter and that there is no Heron triangle having the radius of the circumscribed circle a power of 2 or a power of a prime number p such that $p \equiv 11 \pmod{12}$.

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1. Introduction

A *Heron triangle* is a triangle having the lengths of all its three sides as well as its aria integers. There are still several nice open problems concerning Heron triangles and the book of Richard Guy [5] contains a few such. For example (see D21 in [5]), it is not known if there exist Heron triangles having all medians integers. It is also not known if there exist Pythagorean triples whose products are equal, i.e. if there is a solution of the equation

$$xy(x^4 - y^4) = zw(z^4 - w^4)$$

in nonzero integers $(x,\ y,\ z,\ w)$. Another open problem due to Harborth and Kemnitz (see [6] and [7]) asks to find all Heron triangles whose sides are members of the Fibonacci sequence. It is easy to see that such triangles have to be isosceles but the only known example is the one of sides $(5,\ 5,\ 8)$. Finally, we also mention that another open problem from [5] (see D18) asks for the existence of a perfect cuboid, i.e. a rectangular box with all edges, face diagonals and main diagonal integers. Incidentally, in [8], one of us showed that the existence of a perfect cuboid is equivalent with the existence of a Heron triangle whose sides are perfect squares and whose angle bisectors are rationals but it seems by no means easier to decide whether such a triangle exists. On the positive side of things, we mention that in [9] we found all Heron triangles whose sides are prime powers. Except for the Pythagorean triple $(3,\ 4,\ 5)$ they are in one-to-one correspondence with the Fermat primes larger than 3.

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In what follows, we denote by $a,\ b,\ c$ the sides of a triangle and by A its aria. We also use s, r and R for the semiperimeter, radius of the inscribed circle and radius of the circumscribed circle, respectively. In this note, we collect a few remarks on Heron triangles.

2. Pairs of Heron triangles with the same area

In this section, we point out that there are infinitely many pairs of incongruent Heron triangles having the same area. A particularly pretty parametric family of such pairs of Heron triangles can be obtained using the familiar *Fibonacci sequence.* Recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is the sequence having the initial values $F_0 = 0,\ F_1 = 1$ and satisfying the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

for all integers $n \geq 0$. We have the following result.

Proposition 1. *Let $n \geq 1$ be a positive integer. Then, there exists a pair of incongruent Heron triangles having $A = F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}$.*

Proof of Proposition 1. Let u and v be two positive integers with $u \geq 2$ and $v \geq 1$. Notice that the triangle $T(u, v)$ of sides

$$a = u^2 + v^2$$

$$b = (uv)^2 + 1$$

$$c = (uv)^2 + u^2 - v^2 - 1 \tag{1}$$

has aria

$$A = uv(u^2 - 1)(v^2 + 1). \tag{2}$$

Indeed, formula (2) follows immediately from the well-known formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}. \tag{3}$$

In order to finish the proof it remains to show that one can choose the pairs $(u,\ v)$ in two different ways such that the corresponding triangles $T(u,\ v)$ are incongruent but they have the same area A, namely $F_nF_{n+1}\cdots F_{n+5}$. One can choose the pairs $(u,\ v)$ such that

$$(u,\ v) \in \begin{cases} \{(F_{n+1},\ F_{n+4}),\ (F_{n+2},\ F_{n+3})\} \text{if } n \text{ is even,} \\ \{(F_{n+4},\ F_{n+1}),\ (F_{n+3},\ F_{n+2})\} \quad \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

We check only the case n even as the arguments for the case n odd are similar. Using the well-known formulae

$$F_{n+1}^2 + (-1)^{n+1} = F_n F_{n+2}, \tag{5}$$

and

$$F_{n+2}^2 + (-1)^{n+1} = F_n F_{n+4}, \tag{6}$$

which hold for all integers $n \geq 0$, it follows easily that for n even the area of the triangle T_1 of parameters $(u,\ v) = (F_{n+1},\ F_{n+4})$ is

$$A = F_{n+1}F_{n+4}(F_{n+1}^2 - 1)(F_{n+4}^2 + 1) = F_{n+1}F_{n+4}(F_nF_{n+2})(F_{n+3}F_{n+5}) =$$

$$F_nF_{n+1}\cdots F_{n+5},$$

while the area of the triangle T_2 of parameters $(u,\ v) = (F_{n+2},\ F_{n+3})$ is also

$$A = F_{n+2}F_{n+3}(F_{n+2}^2 - 1)(F_{n+3}^2 + 1) = F_{n+2}F_{n+3}(F_nF_{n+4})(F_{n+1}F_{n+5}) =$$

$$F_nF_{n+1}\cdots F_{n+5}.$$

In order to how that T_1 is incongruent to T_2, it suffices to notice that the shortest side of the triangle given by formula (1) is a. Hence, it is enough to prove that

$$F_{n+1}^2 + F_{n+4}^2 \neq F_{n+2}^2 + F_{n+3}^2. \tag{7}$$

We show that the left side of (7) is larger than the right side of (7). This is equivalent to

$$F_{n+4}^2 - F_{n+3}^2 > F_{n+2}^2 - F_{n+1}^2,$$

or

$$(F_{n+4} - F_{n+3})(F_{n+4} + F_{n+3}) > (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}),$$

or

$$F_{n+2}F_{n+5} > F_nF_{n+3},$$

and this last inequality is obviously true because $F_{n+2} > F_n$ for all integers $n \geq 0$.

Remark 1. At D21 in [5], it is also pointed out that we do not know how many primitive Pythagorean triangles can have the same area. Since every primitive Pythagorean triangle can be parametrized as

$$m^2 + n^2,\ m^2 - n^2,\ 2mn$$

for some coprime positive integers m and n such that $m > n$ and $m \not\equiv n \pmod 2$, this question is equivalent to finding the largest t such that there exist t pairs of generators $(m_i,\ n_i)$ for $i \in \{1,\ 2,\ \ldots,\ t\}$ satisfying the above restrictions and with

$$m_in_i(m_i^2 - n_i^2) = m_jn_j(m_j^2 - n_j^2) \qquad \text{for all } i,\ j \in \{1,\ 2,\ \ldots,\ t\}.$$

For $t = 3$ there are five known examples namely $(77, 38)$, $(78, 55)$, $(138,\ 5)$ found by Shedd 1945, $(1610, 869)$, $(2002, 1817)$, $(2622, 143)$ $(2035, 266)$, $(3306, 61)$, $(3422, 55)$ $(2201, 1166)$, $(2438, 2035)$, $(3565, 198)$ all three found by Rathbun in 1986 and finally $(7238, 2465)$, $(9077, 1122)$, $(10434, 731)$ found in consecutive days by Hoey and Rathbun. It is not known if there are infinitely many examples for $t = 3$ or if there is any example for $t = 4$.

3. More pairs of Heron triangles with the same area and perimeter

We searched for pairs of incongruent Heron triangles having not only the same area A but also the same semiperimeter s. One such example is the pair of triangles of sides $(24,\ 35,\ 53)$ and $(48,\ 14,\ 50)$ both having the same area $A = 336$ and semiperimeter $s = 56$. Based on this example, we found an infinite parametric family of pairs of such triangles. This family is given explicitely as follows.

Proposition 2. *Let $t \geq 1$ be any positive integer and let $T(t)$ and $T_1(t)$ be the triangles of sides*

$$\begin{cases} a := t^8 + 5t^6 + 9t^4 + 7t^2 + 2, \\ b := t^{10} + 5t^8 + 10t^6 + 10t^4 + 6t^2 + 3, \\ c := t^{10} + 6t^8 + 15t^6 + 19t^4 + 11t^2 + 1, \end{cases} \tag{8}$$

and

$$\begin{cases} a_1 := t^{10} + 6t^8 + 14t^6 + 16t^4 + 9t^2 + 2, \\ b_1 := t^6 + 4t^4 + 6t^2 + 3, \\ c_1 := t^{10} + 6t^8 + 15t^6 + 18t^4 + 9t^2 + 1. \end{cases} \tag{9}$$

Then, $T(t)$ and $T_1(t)$ are incongruent Heron triangles having the same semiperimeter namely

$$s = t^{10} + 6t^8 + 15t^6 + 19t^4 + 12t^2 + 3, \tag{10}$$

and the same area, namely

$$A = t(t^2 + 1)^4(t^2 + 2)(t^4 + 3t^2 + 3). \tag{11}$$

Proof of Proposition 2. Rather then checking that the above triangles of sides given by formulae (8) and (9) have indeed semiperimeters and areas given by formulae (10) and (11) we will explain how we found them. For a triangle T of sides $a,\ b,\ c$ and semiperimeter s we let $x = s - a,\ y = s - b$ and $z = s - c$. With these notations, we have $a = y + z,\ b = x + z,\ c = x + y$, $p = x + y + z$

and $A = \sqrt{xyz(x+y+z)}$. For the pairs of triangles $(24,\ 35,\ 53)$ and $(48,\ 14,\ 50)$ mentioned at the beginning of this section we have

$$x = 32 \qquad y = 21, \qquad z = 3, \tag{12}$$

and

$$x_1 = 8, \qquad y_1 = 42, \qquad z_1 = 6. \tag{13}$$

In order to generalize the pattern suggested by formulae (12) and (13), we look for pairs of Heron triangles with the same area and semiperimeter having

$$x = \lambda^n, \qquad y = uv, \qquad z = u, \tag{14}$$

and

$$x_1 = \lambda^m, \qquad y = \lambda^k uv, \qquad z = \lambda^k u, \tag{15}$$

where $u,\ v,\ \lambda,\ m,\ n$ and k are integer valued parameters. Since the two triangles are to have the same semiperimeter and the same area, it follows that $xyz = x_1y_1z_1$. In particular, $m + 2k = n$. Imposing that

$$x + y + z = x_1 + y_1 + z_1,$$

we get

$$\lambda^n + uv + u = \lambda^m + \lambda^k(uv + u),$$

or

$$\lambda^m(\lambda^{n-m} - 1) = (\lambda^k - 1)(uv + u). \tag{16}$$

Since $n - m = 2k$, it follows that equation (16) can be rewritten as

$$\lambda^m(\lambda^{2k} - 1) = (\lambda^k - 1)(uv + u),$$

or

$$\lambda^m(\lambda^k + 1) = u(v + 1). \tag{17}$$

At this point, we can choose $u = \lambda^k + 1$ and $v = \lambda^m - 1$ and formula (17) holds. The last condition that we need to insure is that the common value of the area of the two triangles is indeed an integer. Hence, the number

$$xyz(x + y + z)$$

needs to be a perfect square. Using the preceeding substitutions we get that

$$xyz(x+y+z) = \lambda^n u^2 v(\lambda^n + u(v+1)) = \lambda^n(\lambda^k+1)^2(\lambda^m - 1)(\lambda^n + \lambda^m(\lambda^k+1)) =$$

$$\lambda^{m+n}(\lambda^k+1)^2(\lambda^m-1)(\lambda^{2k}+\lambda^k+1) = \lambda^{2m+2k}(\lambda^k+1)^2(\lambda^m-1)(\lambda^{2k}+\lambda^k+1). \tag{18}$$

In order for the number given by formula (18) to be a perfect square, it suffices to choose λ, k and m such that

$$(\lambda^m - 1)(\lambda^{2k} + \lambda^k + 1) \tag{19}$$

is a perfect square. If we choose $m = 3k$, the number given by formula (19) becomes

$$(\lambda^{3k} - 1)(\lambda^{2k} + \lambda^k + 1) = (\lambda^k - 1)(\lambda^{2k} + \lambda^k + 1)^2. \tag{20}$$

Now the number given by formula (20) is a perfect square when $k = 1$ and $\lambda = t^2+1$ for some positive integer t. Hence, $m = 3$, $n = 5$, $u = \lambda + 1 = t^2 + 2$, and $v = \lambda^3 - 1 = (t^2 + 1)^3 - 1 = t^6 + 3t^4 + 3t^2$. Working backwards we find that the triangles given by formulae (14) and (15) are exactly the ones given by Proposition 2.

Remark 2. We point out that the statement of Proposition 2 is nontrivial in the sens that the pairs $(T(t),\ T_1(t))_{t\geq 1}$ are not similar for different values of t. Indeed, since the two Heron triangles $(24,\ 35,\ 53)$ and $(48,\ 14,\ 50)$ have the same area and semiperimeter, it follows that the two Heron triangles $(24t,\ 35t,\ 53t)$ and $(48t,\ 14t,\ 50t)$ have the same area and semiperimeter as well for any positive integer t. Of course, this is not a very interesting family. To see why the statement of Propositon 2 is non-trivial, notice first that by formula (8), c is always odd. In particular, $\gcd(a,\ b,\ c) = \gcd(x,\ y,\ z)$ for the triangle $T(t)$. But now, by formula (14) and the substitutions from the proof of Proposition 2, one has

$$\gcd(x,\ y,\ z) = \gcd(x,\ u) = \gcd(\lambda^n,\ \lambda^k + 1) = 1.$$

In particular, the Heron triangle $T(t)$ is always primitive. Unfortunately, the triangle $T_1(t)$ is never primitive as it can be immediately noticed from formula (15).

Our Proposition 2 provides an infinite family of pairs of Heron triangles having the same aria and semiperimeter but these are not all of them. For example, the pairs of Heron triangles $\{(51,\ 52,\ 101),\ (17,\ 87,\ 100)\}$, or $\{(20,\ 21,\ 29),\ (17,\ 25,\ 28)\}$ or $\{(17,\ 28,\ 39),\ (12,\ 35,\ 37)\}$ have the same areas and semiperimeters but they are not particular instances of our general family given by formulae (8) and (9). We conclude this section by suggesting the following problem.

Problem. *Find the largest k for which there exist k mutually incongruent Heron triangles having the same area and semiperimeter.*

4. Heron triangles whose area has prescribed prime factors

In this section, we take a look at the prime factors of the area of a Heron triangle.

Let $\mathcal{P}$ be a finite set of primes and let

$$\mathcal{S} := \left\{ n \geq 1 : n = \prod_{p \in \mathcal{P}} p^{\alpha_p} \text{ for some } \alpha_p \geq 0 \right\}. \tag{21}$$

That is, for a fixed finite set of prime numbers $\mathcal{P}$, we let $\mathcal{S}$ be the set of all positive integers whose prime factors belong to $\mathcal{P}$. We first investigate the problem of finding all the Heron triangles having $A \in \mathcal{S}$. We should first notice that if $T := (a,\ b,\ c)$ is a Heron triangle having $A \in \mathcal{S}$, then $d := \gcd(a,\ b,\ c) \in \mathcal{S}$ as well. Moreover, if we let $a = da_1,\ b = db_1$ and $c = dc_1$, then $T_1 := (a_1,\ b_1,\ c_1)$ is a Heron triangle as well and its area $A_1 \in \mathcal{S}$. Hence, it suffices to restrict our attention to primitive Heron triangles $T := (a,\ b,\ c)$. Our first result in this direction is the following.

Proposition 3. *Let $\mathcal{P}$ be a fixed finite set of primes and let $\mathcal{S}$ be given by formula* (21)*. Then, there exist only finitely many primitive Heron triangles having $A \in \mathcal{S}$.*

Proof of Proposition 3. Assume that $T := (a,\ b,\ c)$ satisfies $A \in \mathcal{S}$. Write again $x = s - a,\ y = s - b$ and $z = s - c$. Since T is primitive, it follows that $\gcd(x,\ y,\ z) = 1$. Now the containment $A \in \mathcal{S}$ together with the fact that

$$xyz(x + y + z) = A^2, \tag{22}$$

imply

$$x + y + z \in \mathcal{S}, \qquad \text{where } x,\ y \text{ and } z \in \mathcal{S}. \tag{23}$$

Equation (23) is known as an $\mathcal{S}$-*unit equation* and it has only finitely many solutions satisfying $\gcd(x,\ y,\ z) = 1$ by a result of Evertse (see [4]). Unfortunately,

the proof of Evertse from [4] concerning the finiteness of the number of solutions of equation (23) is not effective. In particular, this means that, apriori, there is no algorithm which for a given $\mathcal{P}$ will allow one to list all primitive Heron triangles having $A \in \mathcal{S}$. It is also known and easy to prove that if A is the area of a Heron triangle, then $6 \mid A$. In particular, if one starts with a set $\mathcal{P}$ of primes and one wants any Heron triangles at all with area $A \in \mathcal{S}$, then one should allow $2,\ 3 \in \mathcal{P}$. In light of these remarks and of Proposition 3, it makes sens to ask if one can determine all the Heron triangles having $A \in \mathcal{S}$ when $\mathcal{P} = \{2,\ 3\}$. Here is the result.

Proposition 4. *Assume that T is a primitive Heron triangle having $S = 2^{\alpha} \cdot 3^{\beta}$ for some non-negative integers α and β. Then, T is congruent to one of the following 10 Heron triangles*

$$(3,\ 4,\ 5),\quad (3,\ 25,\ 26),\quad (4,\ 13,\ 15),\quad (5,\ 5,\ 6),\quad (5,\ 5,\ 8),$$

$$(5,\ 29,\ 30),\quad (9,\ 10,\ 17),\quad (9,\ 73,\ 80),\quad (13,\ 244,\ 255),\quad (17,\ 65,\ 80). \tag{24}$$

Proof of Proposition 4. By equations (22) and (23), it follows that every primitive Heron triangle $T := (a,\ b,\ c)$ having $A = 2^{\alpha} \cdot 3^{\beta}$ can be found by first solving equation

$$x + y + z = w, \tag{25}$$

where $\gcd(x,\ y,\ z) = 1$, the prime factors of x, y and z are in the set $\{2,\ 3\}$ subject to the additional restriction that $xyzw$ is a square. One can easily solve the above equation (even without the restriction that $xyzw$ is a square) using the results of Mo De Ze and R. Tijdeman from [3] (based on Baker's method) in a matter of seconds on a PC, also in the case when the primes $2,\ 3$ are replaced by any pair of primes not exceeding 200. We also point out that equation (25) was solved by elementary means by Leo J. Alex and Lorraine Foster in the series of papers [1] and [2] even when the set of primes $\{2,\ 3\}$ is replaced by the larger set of primes $\{2,\ 3,\ 5\}$. One can now look at all such solutions listed in [1] and [2] and conclude that the only Heron triangles satisfying the hypothesis of Proposition 4 are indeed the ones listed at (24). We omit further details.

Remark 3. In the upcoming paper [10], we show that if $\mathcal{P}$ is a given set of finitely many prime numbers, then there exist only finitely many primitive Heron triangles having $abc \in \mathcal{S}$. Although the full result is not effective, it can be made effective in some instances. For example, in [10], we find all primitive Heron triangles having the property that the maximal prime divisor of abc does not exceed 11.

5. Heron triangles with prescribed r or R

In this section, we leave the issues concerning areas of Heron triangles and we look at existence results for Heron triangles with given integer r or R. Our main results here are the following.

Proposition 5. *Let $k \geq 1$ be a positive integer. Then, there exists a Heron triangle T having $r = k$.*

Proof of Proposition 5. We use the notations $x,\ y,\ z$, etc. from the preceeding sections. Since $r = A/s$, it suffices to show that the equation

$$\frac{xyz}{x + y + z} = k^2 \tag{26}$$

has a positive solution $x,\ y,\ z$. We choose $z = 1$ and equation (26) becomes

$$xy = k^2(x + y + 1),$$

or

$$x(y - k^2) = k^2(y + 1),$$

or

$$x = \frac{k^2(y+1)}{y-k^2}. \tag{27}$$

Clearly, one may now choose $y = k^2 + 1$ and then formula (27) tells us that $x = k^4 + 2k^2$. Hence, the triangle of sides $a = k^2 + 2$, $b = k^4 + 2k^2 + 1$ and $c = k^4 + 3k^2 + 1$ is a Heron triangle with $r = k$.

While Proposition 5 shows that one can construct Heron triangles of arbitrary integer radius r, this is no longer true if one replaces r by R, but it is "almost" true. That is, we have the following result.

Proposition 6. *1. The set of positive integers $k \geq 1$ for which there exists a Heron triangle having $R = k$ has asymptotic density* 1.

2. *There is no Heron triangle having R a power of* 2 *or a power of a prime number p such that $p \equiv 11 \pmod{12}$.*

Proof of Proposition 6, Part 1. Let p be a prime which is congruent to 1 modulo 4. Since p is a sum of two squares, we can write $p = u^2 + v^2$. Then, the triangle of sides

$$a = 2(u^2 + v^2), \qquad b = 2|u^2 - v^2|, \qquad c = 4uv \tag{28}$$

is Heron and has $R = u^2 + v^2 = p$. Indeed, this follows immediately from the fact that the above triangle is right angled, so its R is equal to half of its hypothenuse. It is now clear that if k is an arbitrary positive integer which is a multiple of p, then there exists a Heron triangle of radius $R = k$. To see this, it suffices to consider the triangle which is similar to the triangle given by (28) but whose sides are k/p times longer. The conclusion of 1 follows now from the fact that almost every positive integer is divisible by a prime $p \equiv 1 \pmod 4$.

Before proving Part 2 of Proposition 6, we need to make some considerations concerning arbitrary Heron triangles. Assume that $T := (a,\ b,\ c)$ is a Heron triangle of area A and semiperimeter s and let again $x = s-a$, $y = s-b$, $z = s-c$. Assume also that $D = \gcd(x,\ y,\ z)$ and write $x = Du$, $y = Dv$ and $z = Dw$. The following Lemma turns out to be useful.

Lemma. *(i.)One of the numbers u, v, w is odd and one of them is even. In particular,* $\gcd(a,\ b,\ c) = D$.

(ii.) If p is an odd prime such that $p \mid \gcd(u+v,\ u+w)$, then $p \equiv 1 \pmod 4$.

Proof of the Lemma. (i.) Since $\gcd(u,\ v,\ w) = 1$, it follows that at least one of the numbers u, v, w is odd. We now show that not all of them can be odd. Indeed, since $D \mid \gcd(a,\ b,\ c)$, it follows that the triangle of sides $a_1 = a/D$, $b_1 = b/D$, $c_1 = c/D$ is a Heron triangle of well. Its area A_1 is certainly given by the formula

$$uvw(u+v+w) = A_1^2. \tag{29}$$

Since $6 \mid A_1$, it follows that one of the numbers u, v, w is even. It now follows right away that $\gcd(a,\ b,\ c) = D$.

(ii.) We keep the previous notations. Assume that p is an odd prime number with $p \mid \gcd(u+v,\ u+w) = \gcd(b_1,\ c_1)$. Straightforward computations using the Heron formula for the area A_1 show that formula (29) can be rewritten as

$$-a_1^4 - b_1^4 - c_1^4 + 2a_1^2b_1^2 + 2a_1^2c_1^2 + 2b_1^2c_1^2 = A_1^2. \tag{30}$$

Reducing equation (30) modulo p, we get

$$-a_1^4 \equiv A_1^2 \pmod{p}. \tag{31}$$

Since $\gcd(a_1,\ b_1,\ c_1) = 1$, it follows that $p \not| \ a_1$. Now formula (31) shows that -1 is a quadratic residue modulo p, which implies that $p \equiv 1 \pmod 4$.

Proof of Proposition 6, Part 2. We begin with some general considerations concerning Heron triangles having integer R and then we will specialize to the cases in which R is a power of 2 or a power of a prime number $p \equiv 11 \pmod{12}$. We first deal with a few technicalities due mainly to the fact that we work with arbitrary Heron triangles and not only with primitive ones. We use the formula

$$4RS = abc. \tag{32}$$

We write again $x = s-a,\ y = s-b,\ z = s-c$ and $D = \gcd(x,\ y,\ z)$. Hence, we can write $x = Du,\ y = Dv,\ z = Dw$, where $\gcd(u,\ v,\ w) = 1$. Now formula (32) can be written as

$$16R^2uvw(u+v+w) = D^2(u+v)^2(u+w)^2(v+w)^2. \tag{33}$$

In what follows, for two positive integers s and t, we use sometimes the notation $d_{s,t}$ to designate the greatest common divisor of s and t. Notice that since $\gcd(u,\ v,\ w) = 1$, it follows that

$$\gcd(d_{u,v},\ d_{u,w}) = \gcd(d_{v,u},\ d_{v,w}) = \gcd(d_{w,u},\ d_{w,v}) = 1, \tag{34}$$

and

$$\gcd(d_{u+v,w},\ d_{u,v}d_{u,w}d_{v,w}) = \gcd(d_{u+w,v},\ d_{u,v}d_{u,w}d_{v,w}) = \gcd(d_{v+w,u},\ d_{u,v}d_{u,w}d_{v,w}) = 1. \tag{35}$$

We can now make some cancellations in both sides of formula (33) and get

$$16R^2\frac{u}{d_{u,v}d_{u,w}d_{u,v+w}} \cdot \frac{v}{d_{v,u}d_{v,w}d_{v,u+w}} \cdot \frac{w}{d_{w,u}d_{w,v}d_{w,u+v}} \cdot \frac{u+v+w}{d_{u,v+w}d_{v,u+w}d_{w,u+v}} = D^2\Big(\frac{u+v}{d_{u,v}d_{u+v,w}}\Big)^2 \cdot \Big(\frac{u+w}{d_{u,w}d_{u+w,v}}\Big)^2 \cdot \Big(\frac{v+w}{d_{v,w}d_{v+w,u}}\Big)^2. \tag{36}$$

While formula (36) looks uglier than formula (33), it does have the advantage that it points out that each one of the last three square factors appearing in the right hand side of (36) must divide the factor $16R^2$ from the left hand side of formula (36) (simply because they are coprime with the remaining factors from the laft hand side of formula (36)). In particular, we get that

$$\left(\frac{u+v}{d_{u,v}d_{u+v,w}}\right)\left(\frac{u+w}{d_{u,w}d_{u+w,v}}\right)\left(\frac{v+w}{d_{v,w}d_{v+w,u}}\right) \Bigm| 4R. \tag{37}$$

Let us look at the factors

$$\left(\frac{u+v}{d_{u,v}d_{u+v,w}}\right), \qquad \left(\frac{u+w}{d_{u,w}d_{u+w,v}}\right), \qquad \left(\frac{v+w}{d_{v,w}d_{v+w,u}}\right) \tag{38}$$

from the right hand side of relation (37). By i of the above Lemma, we get that at least two of the numbers of list (38) are odd. Now by ii of the above Lemma, it follows that the greatest common divisor of any two numbers from list (38) is divisible only with primes which are congruent to 1 modulo 4. These considerations show that the only possibilities for the three numbers from list (38) are

$$1,\ 1,\ 2^{\alpha} \qquad \text{for some } \alpha \geq 0, \tag{39}$$

if R is a power of 2 and

$$1,\ 2^{\alpha},\ q, \quad \text{or} \quad 1,\ 1,\ 2^{\alpha}q \quad \text{for some } \alpha \in \{0,\ 1,\ 2\}, \tag{40}$$

if $R = q$ is a power of a prime number p such that $p \equiv 11 \pmod{12}$. At any rate, we may assume that the first number of list (38) is 1. In this case, one can write

$$d_{u,v} = d, \qquad u = d\alpha, \qquad v = d\beta, \qquad w = (\alpha+\beta)\gamma, \tag{41}$$

where $\gcd(\alpha,\ \beta) = 1$. Since $\gcd(u,\ v,\ w) = 1$, it follows easily that

$$d_{u,w} = \gcd(d\alpha,\ (\alpha+\beta)\gamma) = \gcd(\alpha,\ (\alpha+\beta)\gamma) = d_{\alpha,\gamma}, \tag{42}$$

and

$$d_{u+w,v} = \gcd(d\alpha + (\alpha+\beta)\gamma,\ d\beta) = \gcd((d+\gamma)\alpha + \beta\gamma,\ \beta) = d_{d+\gamma,\beta}. \tag{43}$$

Similarly, one can show that

$$d_{v,w} = d_{\beta,\gamma}, \tag{44}$$

and

$$d_{v+w,u} = d_{d+\gamma,\alpha}. \tag{45}$$

With these formulae, we get easily that the second and the third number from list (38) are

$$\frac{u+w}{d_{u,w}d_{u+w,v}} = \frac{d\alpha+(\alpha+\beta)\gamma}{d_{\alpha,\gamma}d_{d+\gamma,\beta}} = \frac{\alpha(d+\gamma)}{d_{\alpha,\gamma}d_{d+\gamma,\beta}} + \frac{\gamma\beta}{d_{\alpha,\gamma}d_{d+\gamma,\beta}}, \tag{46}$$

and

$$\frac{v+w}{d_{v,w}d_{v+w,u}} = \frac{d\beta+(\alpha+\beta)\gamma}{d_{\beta,\gamma}d_{d+\gamma,\alpha}} = \frac{\beta(d+\gamma)}{d_{\beta,\gamma}d_{d+\gamma,\alpha}} + \frac{\gamma\alpha}{d_{\beta,\gamma}d_{d+\gamma,\alpha}}. \tag{47}$$

Formulae (46) and (47) show that each one of the last two numbers of list (38) are strictly larger than 1 (they are each a sum of two positive integers). In particular, this rules out the possibility that R is a power of 2 (compare to list (39)). We shall now only sketch the remaining of the proof of the fact that R cannot be a power of a prime p with $p \equiv 11 \pmod{12}$. Assume that this is not so. By the above considerations and formula (40), it follows that the only possibilities for the numbers from list (38) are

$$1,\ 2,\ q, \quad \text{or} \quad 1,\ 4,\ q. \tag{48}$$

If the second number from list (38) is 2, then we use formula (46) to conclude that

$$\left(\frac{\alpha}{d_{\alpha,\gamma}}\right)\cdot\left(\frac{(d+\gamma)}{d_{d+\gamma,\beta}}\right) = \left(\frac{\gamma}{d_{\alpha,\gamma}}\right)\cdot\left(\frac{\beta}{d_{d+\gamma,\beta}}\right) = 1. \tag{49}$$

Formula (49) implies that $\alpha = \gamma$ and that $\beta = d+\gamma$. The last formulae imply that $d = \beta - \gamma$, therefore

$$u = (\beta-\gamma)\gamma, \qquad v = (\beta-\gamma)\beta, \qquad w = (\gamma+\beta)\gamma, \tag{50}$$

and since $\gcd(u,\ v,\ w) = 1$, it follows that $\gamma \not\equiv \beta \pmod 2$. We now get that

$$v + w = (\beta-\gamma)\beta + (\gamma+\beta)\gamma = \beta^2 + \gamma^2. \tag{51}$$

Moreover,

$$d_{v,w} = \gcd((\beta-\gamma)\beta,\ (\gamma+\beta)\gamma) = 1, \tag{52}$$

and

$$d_{v+w,u} = \gcd(\gamma^2+\beta^2,\ (\beta-\gamma)\gamma) = 1, \tag{53}$$

because $d_{\beta,\gamma} = 1$ and β and γ are incongruent modulo 2. Thus, the last number of list (38) is

$$q = \beta^2 + \gamma^2. \tag{54}$$

Equation (54) is impossible because q is a power of a prime p with $p \equiv 3 \pmod 4$. When the second number of list (38) is 4, then one uses formula (46) to conclude that the only possibilities are

$$\left(\left(\frac{\alpha}{d_{\alpha,\gamma}}\right) \cdot \left(\frac{(d+\gamma)}{d_{d+\gamma,\beta}}\right),\ \left(\frac{\gamma}{d_{\alpha,\gamma}}\right) \cdot \left(\frac{\beta}{d_{d+\gamma,\beta}}\right)\right) \in \{(1,\ 3),\ (2,\ 2),\ (3,\ 1)\}. \quad (55)$$

The case $(2,\ 2)$ above can be ruled out easily by considerations modulo 2. In the remaining two cases, an analysis similar to the one done above when the second number from list (38) was 2, leads to a representation of q of the form $q = \gamma^2 + 3\delta^2$ which is impossible because q is a power of a prime p with $p \equiv 11 \pmod{12}$. We do not give further details. Propostion 6 is therefore completely proved.

Remark 4. The above proof of the second assertion of Proposition 6 does much more than simply prove it. A careful investigation of the arguments employed in it show, for example, that if p is a prime number such that $p \equiv 5 \pmod{12}$, then there exists a unique Heron triangle having $R = p$, which is a Pythagorean triangle. In particular, the triangle $(6,\ 8,\ 10)$ is the unique Heron triangle with $R = 5$.

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Alpar-Vajk Kramer
Cuza Vodă 13
3125 Mediaş, Romania

Florian Luca
Mathematics Department
Bielefeld University
Postfach 10 01 31
33 501 Bielefeld, Germany