

ON POLYNOMIAL VALUES OF THE SUM AND THE PRODUCT OF THE TERMS OF LINEAR RECURRENCES

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Abstract. Let $G^{(i)} = \{G_x^{(i)}\}_{x=0}^\infty$ ($i=1,2,\dots,m$) linear recursive sequences and let $F(x)=dx^q + d_px^p + d_{p-1}x^{p-1} + \dots + d_0$, where d and d_i 's are rational integers, be a polynomial. In this paper we showed that for the equations $\sum_{i=1}^m G_{x_i}^{(i)} = F(x)$ and $\prod_{i=1}^m G_{x_i}^{(i)} = F(x)$ where x_i -s are non-negative integers, with some restriction, there are no solutions in x_i -s and x if $q > q_0$, where q_0 is an effectively computable positive constant.

AMS Classification Number: 11B37

1. Introduction

Let $m \geq 2$ be an integer and define the linear recurrences $G^{(i)} = \{G_x^{(i)}\}_{x=0}^\infty$ ($i = 1, 2, \dots, m$) of order k_i by the recursion

$$(1) \quad G_x^{(i)} = A_1^{(i)} G_{x-1}^{(i)} + A_2^{(i)} G_{x-2}^{(i)} + \dots + A_{k_i}^{(i)} G_{x-k_i}^{(i)} \quad (x \geq k_i \geq 2),$$

where the initial values $G_j^{(i)}$ and the coefficients $A_{j+1}^{(i)}$ ($j = 0, 1, \dots, k_i - 1$) are rational integers. Suppose that

$$A_{k_i}^{(i)} \left(\left| G_0^{(i)} \right| + \left| G_1^{(i)} \right| + \dots + \left| G_{k_i-1}^{(i)} \right| \right) \neq 0$$

for any recurrences and denote the distinct roots of the characteristic polynomial

$$g^{(i)}(u) = u^{k_i} - A_1^{(i)} u^{k_i-1} - \dots - A_{k_i}^{(i)}$$

of the sequence $G^{(i)}$ by $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{t_i}^{(i)}$ ($t_i \geq 2$). It is known that there exist uniquely determined polynomials $p_j^{(i)}(u) \in \mathbf{Q}(\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{t_i}^{(i)})[u]$ ($j = 1, 2, \dots, t_i$) of degree less than the multiplicity $m_j^{(i)}$ of roots $\alpha_j^{(i)}$ such that for $x \geq 0$

$$(2) \quad G_x^{(i)} = p_1^{(i)}(x) \left(\alpha_1^{(i)} \right)^x + p_2^{(i)}(x) \left(\alpha_2^{(i)} \right)^x + \dots + p_{t_i}^{(i)}(x) \left(\alpha_{t_i}^{(i)} \right)^x.$$

Using the terminology of F. Mátyás [9], we say that $G^{(1)}$ is the dominant sequence among the sequences $G^{(i)}$ ($i = 1, 2, \dots, m$) if $m_1^{(1)} = 1$, the polynomial $p_1^{(1)}(x) = a$ is a non-zero constant and, using the notation $\alpha_1^{(1)} = \alpha$,

$$(3) \quad |\alpha| = \left| \alpha_1^{(1)} \right| > \left| \alpha_2^{(1)} \right| \geq \dots \geq \left| \alpha_{t_1}^{(1)} \right| \quad \text{and} \quad |\alpha| \geq \left| \alpha_j^{(i)} \right|$$

for $2 \leq i \leq m$ and $1 \leq j \leq t_i$. (Since $A_{k_1}^{(1)} \in \mathbf{Z} \setminus \{0\}$, therefore $|\alpha| > 1$.) In this case

$$(4) \quad G_x^{(1)} = a\alpha^x + p_2^{(1)}(x) \left(\alpha_2^{(1)} \right)^x + \dots + p_{t_1}^{(1)}(x) \left(\alpha_{t_1}^{(1)} \right)^x.$$

If $\left| \alpha_1^{(i)} \right| > \left| \alpha_j^{(i)} \right|$ ($j = 2, \dots, t_i$) in a sequence $G^{(i)}$ and $m_1^{(i)} = 1$ then we denote $p_1^{(i)}(x)$ by a_i , in the case $i = 1$ by a .

In the following we assume that

$$(5) \quad F(x) = dx^q + d_px^p + d_{p-1}x^{p-1} + \dots + d_0,$$

is a polynomial with rational integer coefficients, where $d \neq 0$, $q \geq 2$ and $q > p$.

In the paper we use the following notations:

$$(6) \quad \Sigma_{x_1, x_2, \dots, x_m} = \sum_{i=1}^m G_{x_i}^{(i)}$$

and

$$(7) \quad \mathcal{G}_{x_1, x_2, \dots, x_m} = \prod_{i=1}^m G_{x_i}^{(i)},$$

where x_i -s are non-negative integers.

The Diophantine equation

$$(8) \quad G_n = F(x)$$

with positive integer variables n and x was investigated by several authors. It is known that if G is a nondegenerate second order linear recurrence, with some restrictions, and $F(x) = dx^q$ then the equation (8) have finitely many integer solutions in variables $n \geq 1$ and $q \geq 2$.

For general linear recurrences we know a similar result (see [11]). A more general result was proved by I. Nemes and A. Pethő [10], furthermore by P. Kiss [4].

Using some other conditions, B. Brindza, K. Liptai and L. Szalay [2] proved that the equation

$$G_{x_1}^{(1)} G_{x_2}^{(2)} = w^q$$

implies that q is bounded above, while L. Szalay [12] made the following generalization of this problem. Let $d \neq 0$ fixed integer and s a product of powers of given primes. Then, under some conditions, the equation $dG_{x_1}^{(1)}G_{x_2}^{(2)} \dots G_{x_m}^{(m)} = sw^q$ in positive integers $w > 1, q, x_1, \dots, x_m$ implies that q is bounded above by a constant.

The author in [8] showed that for the equation $G_n^{(1)}G_m^{(2)} = F(x)$, with some restriction, there are no solutions in n, m and x if $q > q_0$, where q_0 is an effectively computable positive constant.

With some restrictions, P. Kiss and F. Mátyás [7] proved an additive result in this theme, namely, if $\Sigma_{x_1, x_2, \dots, x_m} = sw^q$ for positive integers $w > 1, x_1, x_2, \dots, x_m, q$ and there is a dominant sequence among the sequences $G^{(i)}$, then q is bounded above.

P. Kiss investigated the difference between perfect powers and products or sums of terms of linear recurrences. Such a result is proved in [3] for the sequence $G^{(1)}$ which has the form of (4). Namely, under some restrictions, $|sw^q - G_x^{(1)}| > e^{cx}$ for all integers $w > 1, x, q$ and s , if x and $q > n_1$, where c and n_1 are effectively computable positive numbers. Using some conditions, P. Kiss and F. Mátyás [6] generalized this result substituting $G_x^{(1)}$ by $\prod_{i=1}^m G_{x_i}^{(i)}$, where the sequences $G^{(i)}$ have the form of (4).

F. Mátyás [8] proved a similar result in additive case.

2. Results and proofs

Using the notations mentioned above, we shall prove the following theorems.

Theorem 1. *Let $G^{(i)}$ ($i = 1, 2, \dots, m$; $m \geq 2$) be linear recursive sequences of integers defined by (1). Suppose that $G^{(1)}$ is the dominant recurrence among the sequences $G^{(i)}$ and $\alpha \notin \mathbf{Z}$. Let $K > 1$ and $0 < \delta_1 < 1$ be real constants, $F(x)$ and $\Sigma_{x_1, x_2, \dots, x_m}$ are defined by (5) and (6) with the condition $p < \delta_1 q$. If*

$$x_1 > K \max_{2 \leq i \leq m} (x_i)$$

then the equation

$$(9) \quad \Sigma_{x_1, x_2, \dots, x_m} = F(x),$$

in positive integers $x \geq 2, x_1 > x_2, \dots, x_m, q$ implies that $q < q_1$, where q_1 is an effectively computable number depending on $K, \delta_1, F(x), m$ and the sequences $G^{(i)}$.

Theorem 2. *Let $G^{(i)}$ ($i = 1, 2, \dots, m$; $m \geq 2$) be linear recursive sequences of integers defined by (1). Suppose that $|\alpha_1^{(i)}| > |\alpha_j^{(i)}|$ for $1 \leq i \leq m$ and $2 \leq j \leq t_i$,*

moreover $\alpha_1^{(i)}$ -s are not integers. Let $0 < \gamma < 1$ and $0 < \delta_2 < 1$ be real constants, $F(x)$ and $\mathcal{G}_{x_1, x_2, \dots, x_m}$ are defined by (5) and (7) with the condition $p < \delta_2 q$. If $x_i > \gamma \max(x_1, \dots, x_m)$ for $i = 1, \dots, m$ then the equation

$$(10) \quad \mathcal{G}_{x_1, x_2, \dots, x_m} = F(x),$$

in positive integers $x \geq 2, x_1 > x_2, \dots, x_t, q$ implies that $q < q_2$, where q_2 is an effectively computable number depending on $\gamma, \delta_2, F(x), m$ and the sequences $G^{(i)}$.

Remark. P. Kiss in [5] proved similar results with other conditions.

In what follows we need the following auxiliary results.

Lemma 1. Let $\omega_1, \omega_2, \dots, \omega_n$ ($\omega_i \neq 0$ or 1) be algebraic numbers with heights at most $M_1, M_2, \dots, M_n \geq 4$, respectively. If b_1, b_2, \dots, b_n are non-zero integers with $\max(|b_1|, |b_2|, \dots, |b_{n-1}|) \leq B$ and $|b_n| \leq B'$, $B' \geq 3$, furthermore

$$\Lambda = |b_1 \log \omega_1 + b_2 \log \omega_2 + \dots + b_n \log \omega_n| \neq 0,$$

where the logarithms are assumed to have their principal values, then there exists an effectively computable positive constant C , depending only on n, M_1, \dots, M_{n-1} and the degree of the field $\mathbf{Q}(\omega_1, \dots, \omega_n)$ such that

$$\Lambda > \exp(-C \log B' \log M_n - B/B').$$

Lemma 1. is a result of A. Baker (see Theorem 1. in [1] with $\delta = 1/B'$).

For the sake of brevity we introduce the following abbreviations. For non-negative integers x_1, x_2, \dots, x_m let

$$(11) \quad \varepsilon_j^{(i)} = \frac{p_j^{(i)}(x_i) \left(\alpha_j^{(i)}\right)^{x_i}}{a \alpha^{x_1}}, \quad \varepsilon_1 = \sum_{j=2}^{t_1} \varepsilon_j^{(1)}, \quad \varepsilon_2 = \sum_{i=2}^m \sum_{j=1}^{t_i} \varepsilon_j^{(i)}$$

and $\varepsilon = \varepsilon_1 + \varepsilon_2$. Using (2), (4) and (6)

$$\Sigma_{x_1, x_2, \dots, x_m} = a\alpha^{x_1} + \sum_{j=2}^{t_1} p_j^{(1)}(x_1) \left(\alpha_j^{(1)}\right)^{x_1} + \sum_{i=2}^m \sum_{j=1}^{t_i} p_j^{(i)}(x_i) \left(\alpha_j^{(i)}\right)^{x_i}$$

and by (11) we have

$$(12) \quad \Sigma_{x_1, x_2, \dots, x_m} = a\alpha^{x_1} (1 + \varepsilon_1 + \varepsilon_2) = a\alpha^{x_1} (1 + \varepsilon).$$

Let

$$(13) \quad \varepsilon_3 = \left(\frac{d_p}{d} \left(\frac{1}{x} \right)^{q-p} \right) \left(1 + \frac{d_{p-1}}{d_p} \left(\frac{1}{x} \right) + \dots + \frac{d_0}{d_p} \left(\frac{1}{x} \right)^p \right).$$

So (5) can be written in the form

$$(14) \quad F(x) = dx^q(1 + \varepsilon_3).$$

The following three lemmas are due to F. Mátyás [8], where n_1, n_2, n_3 means effectively computable constants.

Lemma 2. *Let $G^{(1)}$ be the dominant sequence among the recurrences $G^{(i)}$ ($1 \leq i \leq m$) defined by (1). Then there are effectively computable positive constants c_1 and n_1 depending only on the sequence $G^{(1)}$ such that*

$$|\varepsilon_1| < e^{-c_1 x_1}$$

for any $n_1 < x_1$.

Lemma 3. *Let $G^{(1)}$ be the dominant sequence among the recurrences $G^{(i)}$ ($1 \leq i \leq m$) defined by (1), $1 < K \in \mathbf{R}$ and $x_1 > K \max_{2 \leq i \leq m} (x_i)$. Then there are effectively computable positive constants c_2 and n_2 depending only on K and the sequences $G^{(i)}$ such that*

$$|\varepsilon_2| < e^{-c_2 x_1}$$

for any $n_2 < x_1$.

Lemma 4. *Suppose that the conditions of Lemma 2 and Lemma 3 hold. Then there exist effectively computable positive constants c_3, c_4, n_3 depending only on K and the sequences $G^{(i)}$ such that*

$$e^{c_3 x_1} < |\Sigma_{x_1, x_2, \dots, x_m}| < e^{c_4 x_1}$$

for any integer $x_1 > n_3$.

The following lemma is due to P. Kiss and F. Mátyás [6].

Lemma 5. *Let γ be a real number with $0 < \gamma < 1$ and let $\mathcal{G}_{x_1, \dots, x_m}$ be an integer defined by (7), where x_1, \dots, x_m are positive integers satisfying the condition $x_i > \gamma \max(x_1, \dots, x_t)$ and $|\alpha_1^{(i)}| > |\alpha_j^{(i)}|$ for $1 \leq i \leq m$ and $2 \leq j \leq t_i$. Then there are effectively computable positive constants c_5 and n_4 , depending only on the sequences $G^{(i)}$ and γ , such that*

$$(15) \quad \mathcal{G}_{x_1, \dots, x_m} = \left(\prod_{i=1}^m a_i \alpha_i^{x_i} \right) (1 + \varepsilon_4),$$

where

$$|\varepsilon_4| < e^{-c_5 \cdot \max(x_1, \dots, x_m)}$$

for any $\max(x_1, \dots, x_m) > n_4$.

Remark. In general $\alpha_1^{(i)}$ is named the dominant root of the i -th sequence, if $\left| \alpha_1^{(i)} \right| > \left| \alpha_j^{(i)} \right|$ for $2 \leq j \leq t_i$.

Proof of Theorem 1. In the proof c_6, c_7, \dots denote effectively computable constants, which depend on $K, \delta_1, F(x)$ and the sequences $G^{(i)}$. Suppose that (9) holds with the conditions given in the Theorem 1. and x_1 is sufficiently large. Using (9), (14) and Lemma 4. we have

$$(16) \quad |dx^q(1 + \varepsilon_3)| = |F(x)| = |\Sigma_{x_1, x_2, \dots, x_m}| < e^{c_6 x_1}.$$

Taking the logarithms of the both side we get

$$|\log |d| + q \log x + \log |1 + \varepsilon_3|| < c_6 x_1$$

that is

$$(17) \quad q \log x < c_7 x_1.$$

Now, using (11) and (13), the equation (9) can be written in the form

$$(18) \quad \left| \frac{a\alpha^{x_1}}{dx^q} \right| = |1 + \varepsilon_3| |1 + \varepsilon|^{-1}.$$

We distinguish two cases. First we suppose that

$$a\alpha^{x_1} = dx^q.$$

Let $\alpha' \neq \alpha$ be any conjugate of α and let φ be an automorphism of $\overline{\mathbf{Q}}$ with $\varphi(\alpha) = \alpha'$. Moreover,

$$\varphi(a) (\varphi(\alpha))^{x_1} = \varphi(dx^q).$$

Thus

$$\frac{a}{\varphi(a)} = \left(\frac{\alpha'}{\alpha} \right)^{x_1}.$$

whence x_1 is bounded, which implies that q is bounded. Now we can suppose that $\frac{a\alpha^{x_1}}{dx^q} \neq 1$. Put

$$L_1 = \left| \log \left| \frac{a\alpha^{x_1}}{dx^q} \right| \right| = \left| \log |a| + x_1 \log |\alpha| - q \log x - \log d \right|$$

and employ Lemma 1. with $M_4 = x, B' = q$ and $B = x_1$. We have

$$(19) \quad L_1 > \exp(-c_8 \log q \log x - \frac{x_1}{q}).$$

Using (9), (11), (12), (13), (14) and (17) we have

$$c_9 x^q < dx^q(1 + \varepsilon_3) = a\alpha^{x_1}(1 + \varepsilon_1 + \varepsilon_2) < c_{10} x^q,$$

that is

$$c_{11} x^q < \alpha^{x_1} < c_{12} x^q.$$

Using (13), the previous inequalities and the condition $p < \delta_1 q$ we have

$$(20) \quad |\varepsilon_3| < \left(\frac{1}{x}\right)^{c_{13}(q-p)} < \left(\frac{1}{x}\right)^{c_{13}q(1-\delta_1)} < \exp(-c_{14}x_1).$$

Recalling that $|\log(1+x)| \leq x$ and $|\log(1-x)| \leq 2x$ for $0 \leq x < \frac{1}{2}$ and using (20), Lemma 2. and Lemma 3. we find that

$$\left| \log |1 + \varepsilon_3| |1 + \varepsilon|^{-1} \right| < \exp(-c_{15}x_1)$$

Using (17), (18), (19) and (20) we have the following inequalities

$$c_{15}x_1 < c_8 \log q \log x + \frac{x_1}{q} < c_8 \log q \frac{c_7 x_1}{q} + \frac{x_1}{q} < c_{16}x_1 \frac{\log q}{q}.$$

This implies

$$\frac{c_{15}}{c_{16}} < \frac{\log q}{q}.$$

The previous inequality can be satisfied by only finitely many q and this completes the proof.

Proof of Theorem 2. Similarly the previous proof, c_i -s denote effectively computable positive constants, which depend on $\gamma, \delta_2, F(x)$ and the sequences $G^{(i)}$. Suppose that (10) holds with the conditions given in Theorem 2. Let x_1, \dots, x_t be positive integers and let $x_0 = \max(x_1, \dots, x_t)$. We suppose that α_s is the dominant root of the sequence which belongs to x_0 . Using Lemma 5. we have

$$(21) \quad e^{c_{17}x_0} < \mathcal{G}_{x_1, \dots, x_m} = F(x) < e^{c_{18}x_0}$$

if $x_0 > n_4$. So by (10) and (21) we get

$$(22) \quad |dx^q(1 + \varepsilon_3)| = |F(x)| = |\mathcal{G}_{x_1, x_2, \dots, x_m}| < e^{c_{18}x_0}.$$

Taking the logarithms of the both side we get

$$|\log |d| + q \log x + \log |1 + \varepsilon_3|| < c_{18}x_0$$

that is

$$(23) \quad q \log x < c_{19} x_0.$$

The equation (10) can be written in the form

$$(24) \quad \frac{\prod_{i=1}^m a_i \alpha_i^{x_i}}{dx^q} = (1 + \varepsilon_3)(1 + \varepsilon_4)^{-1}.$$

We distinguish two cases. First we suppose that

$$\prod_{i=1}^m a_i \alpha_i^{x_i} = dx^q.$$

Let $\alpha'_s \neq \alpha_s$ be any conjugate of α_s and let φ be an automorphism of $\overline{\mathbf{Q}}$ with $\varphi(\alpha) = \alpha'$. Moreover,

$$\varphi \left(\prod_{i=1}^m a_i \alpha_i^{x_i} \right) = \varphi(dx^q)$$

that is

$$\prod_{i=1}^m a_i \alpha_i^{x_i} = \varphi \left(\prod_{i=1}^m a_i \alpha_i^{x_i} \right).$$

Since α dominant root, $\varphi(\alpha_i) \leq \alpha_i$ $i = 1, 2, \dots, m$ we have

$$\left(\frac{\alpha_s}{\varphi(\alpha_s)} \right)^{x_0} \leq \frac{\varphi \left(\prod_{i=1}^m a_i \right)}{\prod_{i=1}^m a_i},$$

whence x_0 is bounded, which implies that q is bounded. Now we can suppose that

$\prod_{i=1}^t a_i \alpha_i^{x_i} \neq dx^q$. Put

$$L_2 = \left| \log \left| \frac{\prod_{i=1}^m a_i \alpha_i^{x_i}}{dx^q} \right| \right| = \left| \sum_{i=1}^m \log |a_i| + \sum_{i=1}^m x_i \log |\alpha_i| - \log d - q \log x \right|$$

and employ Lemma 1. with $M_{2t+2} = x, B' = q$ and $B = x_0$. We have

$$(25) \quad L_2 > \exp(-c_{20} \log q \log x - \frac{x_0}{q}).$$

Using (15) and Lemma 5. we have

$$c_{20}x^q < dx^q(1 + \varepsilon_3) = \prod_{i=1}^m a_i \alpha_i^{x_i} (1 + \varepsilon_4) < c_{21}x^q$$

that is

$$\alpha_s^{x_0} < c_{21}x^q.$$

Using (13), the previous inequality and the condition $p < \delta_2 q$ we have

$$(26) \quad |\varepsilon_3| < \left(\frac{1}{x}\right)^{c_{22}(q-p)} < \left(\frac{1}{x}\right)^{c_{22}q(1-\delta_2)} < \exp(-c_{23}x_0).$$

Recalling that $|\log(1+x)| \leq x$ and $|\log(1-x)| \leq 2x$ for $0 \leq x < \frac{1}{2}$ and using (26) and Lemma 5. we find that

$$(27) \quad \left| \log |1 + \varepsilon_3| |1 + \varepsilon_4|^{-1} \right| < \exp(-c_{24}x_0).$$

Using (23), (24), (25), and (27) we have the following inequalities

$$c_{24}x_0 < c_{20} \log q \log x + \frac{x_0}{q} < c_{20} \log q \frac{c_{19}x_0}{q} + \frac{x_0}{q} < c_{25}x_0 \frac{\log q}{q}.$$

This implies

$$\frac{c_{24}}{c_{25}} < \frac{\log q}{q}.$$

The previous inequality can be satisfied by only finitely many q and this completes the proof.

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