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THE LIE AUGMENTATION TERMINALS OF GROUPS

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Abstract. In this paper we give necessary and sufficient conditions for groups which have finite Lie terminals with respect to commutative ring of non-zero characteristic m, where m is a composite number.

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1. Introduction

Let R be a commutative ring with identity, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the ring homomorphism $\phi : RG \to R$ which maps the group elements to 1. It is easy to see that as R-module $A(RG)$ is a free module with the elements $g-1\,(g \in G)$ as a basis. It is clear that $A(RG)$ is the ideal generated by all elements of the form $g-1$ $(g \in G)$.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively:
$A(RG) = A^{[1]}(RG)$, $A^{[\lambda+1]}(RG) = [A^{[\lambda]}(RG), A(RG)]\cdot RG$, if λ is not a limit ordinal, and $A^{[\lambda]}(RG) = \bigcap\limits_{\nu<\lambda} A^{[\nu]}(RG)$ otherwise, where $[K, M]$ denotes the $R-$submodule of RG generated by $[k, m] = km - mk, k \in K, m \in M$, and for $K \subseteq RG, K\cdot RG$ denotes the right ideal generated by K in RG (similarly $RG\cdot K$ will denote the left ideal generated by K). It is easy to see that the right ideal $A^{[\lambda]}(RG)$ is a two-sided ideal of RG for all ordinals $\lambda \geq 1$. We have the following sequence

$$A(RG) \supseteq A^2(RG) \supseteq \dots$$

of ideals of RG. Evidently there exists the least ordinal $\tau = \tau_R[G]$ such that $A^{[\tau]}(RG) = A^{[\tau+1]}(RG)$ which is called the *Lie augmentation terminal* (or *Lie terminal* for simple) of G with respect to R.

In this paper we give necessary and sufficient conditions for groups which have finite Lie terminal with respect to a commutative ring of non-zero characteristic.

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2. Notations and some known facts

If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short) denotes the ideal of RG generated by all elements of the form $h-1$ $(h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi} : RG \rightarrow RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

Let F be a free group on the free generators $x_i (i \in I)$, and ZF be its integral group ring (Z denotes the ring of rational integers). Then every homomorphism $\phi : F \rightarrow G$ induces a ring homomorphism $\overline{\phi} : ZF \rightarrow RG$ by letting $\overline{\phi}(\sum n_y y) = \sum n_y \phi(y)$, where $y \in F$ and the sum runs over the finite set of $n_y y \in ZF$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of RG generated by the elements $\overline{\phi}(f)$, $\phi \in \mathrm{Hom}(F, G)$, the set of homomorphism from F to G. In other words $A_f(RG)$ is the ideal generated by the values of f in RG as the elements of G are substituted for the free generators x_i-s.

An ideal J of RG is called a *polynomial ideal* if $J = A_f(RG)$ for some $f \in ZF$, F a free group.

It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an $R-$module by the elements $g-1$ $(g \in G)$, i.e. by the values of the polynomial $x-1$.

Lemma 2.1. ([2], Corollary 1.9, page 6.) *The Lie powers $A^{[n]}(RG)(n \geq 1)$ are polynomial ideals in RG.*

We use the following lemma, too.

Lemma 2.2. ([2] Proposition 1.4, page 2.) *If $f \in ZF$, then f defines a polynomial ideal $A_f(RG)$ in every group ring RG. Further, if $\theta : RG \rightarrow KH$ is a ring homomorphism induced by a group homomorphism $\phi : G \rightarrow H$ and a ring homomorphism $\psi : R \rightarrow K$, then*

$$\theta(A_f(RG)) \subseteq A_f(KH).$$

(It is assumed that $\psi(1_R) = 1_K$, where 1_R and 1_K are identity of the rings R and K, respectively.)

Let $\overline{\theta} : RG \rightarrow R/LG$ be an epimorphism induced by the ring homomorphism θ of R onto R/L. By Lemma 2.1 $A^{[n]}(RG)(n \geq 1)$ are polynomial ideal and from Lemma 2.2 it follows that

$$\overline{\theta}(A^{[n]}(RG)) = A^{[n]}(R/LG). \tag{1}$$

Let p be a prime and n a natural number. In this case let's denote by G^{p^n} the subgroup generated by all elements of the form g^{p^n} $(g \in G)$.

If K, L are two subgroups of G, then we denote by (K, L) the subgroup generated by all commutators $(g, h) = g^{-1}h^{-1}gh$, $g \in K$, $h \in L$.

The n^{th} term of the lower central series of G is defined inductively: $\gamma_1(G) = G, \gamma_2(G) = G'$ is the derived group (G, G) of G, and $\gamma_n(G) = (\gamma_{n-1}(G), G)$. The normal subgroups $G_{p,k}$ $(k = 1, 2, \dots)$ is defined by

$$G_{p,k} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G).$$

We have the following sequence of normal subgroups $G_{p,k}$ of a group G

$$G = G_{p,1} \supseteq G_{p,2} \supseteq \dots \supseteq G_p,$$

where $G_p = \bigcap\limits_{k=1}^{\infty} G_{p,k}$.

In [1] the following theorem was proved.

Theorem 2.1. *Let R be a commutative ring with identity of characteristic p^n, where p a prime number. Then*

1. $\tau_R[G] = 1$ *if and only if* $G = G_p$,
2. $\tau_R[G] = 2$ *if and only if* $G \neq G' = G_p$,
3. $\tau_R[G] > 2$ *if and only if* G/G_p *is a nilpotent group whose derived group is a finite p-group.*

3. The Lie augmentation terminal

It is clear, that if G is an Abelian group, then $A^{[2]}(RG) = 0$. Therefore we may assume that the derived group $G' = \gamma_2(G)$ of G is non-trivial.

We considere the case $\operatorname{char} R = m = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s} (s \geq 1)$. Let $\Pi(m) = \{p_1, p_2, \dots, p_s\}$ and $R_{p_i} = R/p_i^{n_i}R$ $(p_i \in \Pi(m))$. If $\overline{\theta}$ is the homomorphism of RG onto $R_{p_i}G$, then by (1)

$$\overline{\theta}(A^{[n]}(RG)) = A^{[n]}(R_{p_i}G) \tag{2}$$

and

$$A^{[n]}(R_{p_i}G) \cong (A^{[n]}(RG) + p_i^{n_i}RG)/p_i^{n_i}RG. \tag{3}$$

Theorem 3.1. *Let G be a non-Abelian group and R be a commutative ring with identity of non-zero characteristic $m = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s} (s \geq 1)$ Then the Lie augmentation terminal of G with respect to R is finite if and onli if for every $p_i \in \Pi(m)$ one of the following conditions holds:*

1. $G = G_{p_i}$
2. $G \neq G' = G_{p_i}$
3. G/G_{p_i} is a nilpotent group whose derived group is a finite p_i-group.

Proof. Let $p_i \in \Pi(m)$ and let one of the conditions hold: $G = G_{p_i}$ or $G \neq G' = G_{p_i}$ or G/G_{p_i} is a nilpotent group whose derived group is a finite p_i-group. From (2),(3) and Theorem 2.1 it follows, that for every $p_i \in \Pi(m)$ there exists $k_i \geq 1$ such that

$$A^{[k_i]}(R_{p_i}G) = A^{[k_i+1]}(R_{p_i}G) = \dots,$$

where $R_{p_i} = R/p_i^{n_i}R$. If

$$k = \max_{i=1}^{s}\{k_i\},$$

then

$$A^{[k]}(R_{p_i}G) = A^{[k+1]}(R_{p_i}G) = \dots$$

for all $p_i \in \Pi(m)$.

Since $A^{[n]}(R_{p_i}G) \cong (A^{[n]}(RG) + p_i^{n_i}RG)/p_i^{n_i}RG$ for all n and every $p_i \in \Pi(m)$, then from the previous isomorphism it follows, that an arbitrary element $x \in A^{[k]}(RG)$ can be written as

$$x = x_i + p_i^{n_i}a_i,$$

where $x_i \in A^{[k+1]}(RG), a_i \in RG$. If $m_i = m/p_i^{n_i}$, then $m_i x = m_i x_i$ since $m_i p_i^{n_i}$ is zero in R. We have

$$\left(\sum_{p_i \in \Pi(m)} m_i\right) x = \sum_{p_i \in \Pi(m)} m_i x_i.$$

Obviously m_i and $p_i^{n_i}$ are coprime numbers and for all $p_i \in \Pi(m)$ $p_i^{n_i}$ divides m_j for $j \neq i$. Therefore $\sum_{p_i \in \Pi(m)} m_i$ and the characteristic m of the ring R are coprime numbers. Consequently $\sum_{p_i \in \Pi(m)} m_i$ is invertible in R. So

$$x = a \sum_{p_i \in \Pi(m)} m_i x_i,$$

where $a \sum_{p_i \in \Pi(m)} m_i = 1$. Hence $x \in A^{[k+1]}(RG)$ and $x \in A^{[k]}(RG) = A^{[k+1]}(RG)$.

Conversely. Let $\tau_R(G) = n \geq 1$, i.e. $A^{n-1}(RG) \neq A^n(RG) = A^{n+1}(RG) = \dots$. Then for every prime $p_i \in \Pi(m)$

$$A^{k-1} \neq A^{[k]}(R_{p_i}G) = A^{[k+1]}(R_{p_i}G) = \dots$$

holds for a suitable $k \leq n$ and Theorem 2.1 completes the proof.

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