

**THE CONDITION FOR GENERALIZING INVERTIBLE  
SUBSPACES IN CLIFFORD ALGEBRAS**

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**Abstract.** Let  $\mathcal{A}$  be a universal Clifford algebra induced by  $m$ -dimensional real linear space with basis  $\{e_1, e_2, \dots, e_m\}$ . The necessary and sufficient condition for the subspaces of form  $L_1 = \text{lin}\{e_0, e_1, \dots, e_m, e_{m+1}, \dots, e_{m+s}\}$  to be invertible is  $m \equiv 2 \pmod{4}$ ,  $s=1$  and  $e_{m+1} = e_{12\dots m}$  (see [2]). In this paper we improve this assertion for the subspaces of the form  $L = \text{lin}\{e_0, e_{A_1}, \dots, e_{A_m}, e_{A_{m+1}}, \dots, e_{A_{m+s}}\}$ , where  $A_i \subseteq \{1, 2, \dots, m\}$  ( $i=1, 2, \dots, m+s$ ).

**1. Introduction**

Let  $V_m$  be an  $m$ -dimensional ( $m \geq 1$ ) real linear space with basis  $\{e_1, e_2, \dots, e_m\}$ . Consider the  $2^m$ -dimensional real space  $\mathcal{A}$  with basis

$$E = \{e_\emptyset, e_{\{1\}}, \dots, e_{\{m\}}, e_{\{1,2\}}, \dots, e_{\{m-1,m\}}, \dots, e_{\{1,2,\dots,m\}}\},$$

where  $e_{\{i\}} := e_i$  ( $i = 1, 2, \dots, m$ ).

In the following, for each  $K = \{k_1, k_2, \dots, k_t\} \subseteq \{1, 2, \dots, m\}$  we write  $e_K = e_{k_1 k_2 \dots k_t}$  with  $e_\emptyset = e_0$ , and so

$$E = \{e_0, e_1, \dots, e_m, e_{12}, \dots, e_{m-1m}, \dots, e_{12\dots m}\}.$$

The product of two elements  $e_A, e_B \in E$  is given by

$$(1) \quad e_A e_B = (-1)^{\#\{A \cap B\}} (-1)^{p(A,B)} e_{A \Delta B}; \quad A, B \subseteq \{1, 2, \dots, m\},$$

where

$$\begin{cases} p(A, B) = \sum_{j \in B} p(A, j). \\ p(A, j) = \#\{i \in A : i > j\}, \\ A \Delta B = (A \setminus B) \cup (B \setminus A) \end{cases}$$

and  $\#A$  denotes the number of elements of  $A$ .

Each element  $a = \sum_A a_A e_A \in \mathcal{A}$  is called a Clifford number. The product of two Clifford numbers  $a = \sum_A a_A e_A$ ;  $b = \sum_B b_B e_B$  is defined by the formula

$$ab = \sum_A \sum_B a_A b_B e_A e_B.$$

It is easy to check that in this way  $\mathcal{A}$  is turned into a linear associative non-commutative algebra over  $\mathbf{R}$ . It is called the Clifford algebra over  $V_m$ .

It follows at once from the multiplication rule (1) that  $e_\emptyset$  is identity element, which is denoted by  $e_0$  and in particular

$$e_i e_j + e_j e_i = 0 \text{ for } i \neq j; \quad e_j^2 = -1 \quad (i, j = 1, 2, \dots, m)$$

and

$$e_{k_1 k_2 \dots k_t} = e_{k_1} e_{k_2} \dots e_{k_t}; \quad 1 \leq k_1 < k_2 < \dots < k_t \leq m.$$

The involution for basic vectors is given by

$$\bar{e}_{k_1 k_2 \dots k_t} = (-1)^{\frac{t(t+1)}{2}} e_{k_1 k_2 \dots k_t}.$$

For any  $a = \sum_A a_A e_A \in \mathcal{A}$ , we write  $\bar{a} = \sum_A a_A \bar{e}_A$ . For any Clifford number  $a = \sum_A a_A e_A$ , we write  $|a| = \left( \sum_A a_A^2 \right)^{\frac{1}{2}}$ .

## 2. Result and Proof

We use the following definitions.

(i) An element  $a \in \mathcal{A}$  is said to be invertible if there exists an element  $a^{-1}$  such that  $aa^{-1} = a^{-1}a = e_0$ ;  $a^{-1}$  is said to be the inverse of  $a$ .

(ii) A subspace  $X \subset \mathcal{A}$  is said to be invertible if every non-zero element in  $X$  is invertible in  $\mathcal{A}$ .

(iii)  $L(u_1, u_2, \dots, u_n) = \text{lin}\{u_1, u_2, \dots, u_n\}$ ,  $u_i \in \mathcal{A}$  ( $i = 1, 2, \dots, n$ ).

It is well-known (see [1]) that for any special Clifford number of the form  $a = \sum_{i=0}^m a_i e_i \neq 0$  we have  $a^{-1} = \frac{\bar{a}}{|a|^2}$ . So  $L(e_0, e_1, \dots, e_m)$  is invertible, and if  $m \equiv 2$

(mod 4) (see [2]), then every  $a = \sum_{i=0}^{m+1} a_i e_i \neq 0$ , where  $e_{m+1} = e_{12\dots m}$  is invertible

and  $a^{-1} = \frac{\bar{a}}{|a|^2}$ . So  $L(e_0, e_1, \dots, e_m, e_{m+1})$  is invertible.

We shall need the following lemmas.

**Lemma 1.** (see Lemma 1 [3]) *If  $L(e_{A_1}, e_{A_2}, \dots, e_{A_k})$ , where  $e_{A_i} \in E, e_{A_i} \neq e_{A_j}$  for  $i \neq j, i, j \in \{1, 2, \dots, k\}$ , is invertible if and only if  $L(e_{A_1}\bar{e}_{A_k}, e_{A_2}\bar{e}_{A_k}, \dots, e_{A_k}\bar{e}_{A_k})$  is invertible.*

By Lemma 1 we shall study subspaces of  $\mathcal{A}$  in the form  $L(e_0, e_{A_1}, \dots, e_{A_l})$ .

**Lemma 2.** (see Lemma 3 [3])  *$L(e_0, e_{A_1}, \dots, e_{A_l}), e_{A_i} \in E, e_{A_i} \neq e_{A_j}$  for  $i \neq j$ , is invertible if and only if*

$$e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0 \text{ for } i \neq j, \quad i, j \in \{0, 1, 2, \dots, m\}, \text{ where } e_{A_0} = e_0.$$

**Lemma 3.** (see Theorem [3]) *If  $L(e_0, e_{A_1}, e_{A_2}, \dots, e_{A_l}), e_{A_i} \in E, e_{A_i} \neq e_{A_j}$  for  $i \neq j, i, j \in \{1, 2, \dots, l\}$  is invertible, then*

(i)  $l \leq m + 1$ .

(ii) *If  $l = m + 1$ , then*

$$\text{either } e_{A_l} = e_{A_1}e_{A_2} \dots e_{A_{l-1}} \quad \text{or} \quad e_{A_l} = -e_{A_1}e_{A_2} \dots e_{A_{l-1}}.$$

The purpose of this paper is to prove the following.

**Theorem.**  *$L(e_0, e_{A_1}, \dots, e_{A_m}, e_{A_{m+1}}, \dots, e_{A_{m+s}})$  is invertible if and only if the following conditions simultaneously hold:*

(1)  $e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0$  for  $i \neq j, i, j \in \{0, 1, 2, \dots, m\}$ , where  $e_{A_0} = e_0$ ,

(2)  $m \equiv 2 \pmod{4}$ ,

(3)  $s = 1$ ,

(4) *Either*  $e_{A_{m+1}} = e_{A_1}e_{A_2} \dots e_{A_m}$  *or*  $e_{A_{m+1}} = -e_{A_1}e_{A_2} \dots e_{A_m}$ .

**Proof.** First we prove the sufficiency. From  $e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0$  for  $i \neq j, i, j \in \{0, 1, \dots, m\}$  we have

$$\bar{e}_{A_i}\bar{e}_{A_j} + \bar{e}_{A_j}\bar{e}_{A_i} = 0 \text{ and } e_{A_i} + \bar{e}_{A_i} = 0 \text{ for } i \neq j, \quad i, j \in \{1, \dots, m\}.$$

We shall prove that  $e_{A_k}\bar{e}_{A_{m+1}} + e_{A_{m+1}}\bar{e}_{A_k} = 0$  for  $k \in \{0, 1, \dots, m\}$ . For  $k = 0$ , by  $\overline{ab} = \bar{b}\bar{a}$  and by  $m \equiv 2 \pmod{4}$ , we get that

$$\begin{aligned} e_0\bar{e}_{A_{m+1}} + e_{A_{m+1}}\bar{e}_0 &= \overline{e_{A_1}e_{A_2} \dots e_{A_m}} + e_{A_1}e_{A_2} \dots e_{A_m} \\ &= \bar{e}_{A_m}\bar{e}_{A_{m-1}} \dots \bar{e}_{A_1} + e_{A_1}e_{A_2} \dots e_{A_m} \\ &= (-1)^m e_{A_m}e_{A_{m-1}} \dots e_{A_1} + e_{A_1}e_{A_2} \dots e_{A_m} \end{aligned}$$

$$\begin{aligned}
&= (-1)^m (-1)^{\frac{m(m-1)}{2}} e_{A_1} e_{A_2} \dots e_{A_m} + e_{A_1} e_{A_2} \dots e_{A_m} \\
&= -e_{A_1} e_{A_2} \dots e_{A_m} + e_{A_1} e_{A_2} \dots e_{A_m} = 0.
\end{aligned}$$

For  $k \in \{1, 2, \dots, m\}$  we have

$$\begin{aligned}
&e_{A_k} \bar{e}_{A_{m+1}} + e_{A_{m+1}} \bar{e}_{A_k} = e_{A_k} \bar{e}_{A_m} \dots \bar{e}_{A_k} \dots \bar{e}_{A_1} + e_{A_1} \dots e_{A_k} \dots e_{A_m} \bar{e}_{A_k} \\
&= (-1)^{m-k} \bar{e}_{A_m} \dots e_{A_k} \bar{e}_{A_k} \dots \bar{e}_{A_1} + (-1)^{m-k} e_{A_1} \dots e_{A_k} \bar{e}_{A_k} \dots e_{A_m} \\
&= (-1)^{m-k} \left[ (-1)^{m-1} e_{A_m} \dots e_{A_{k+1}} e_{A_{k-1}} \dots e_{A_1} + e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m} \right] \\
&= (-1)^{m-k} \left[ -(-1)^{\frac{(m-1)(m-2)}{2}} e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m} \right. \\
&\quad \left. + e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m} \right] = 0.
\end{aligned}$$

Take  $0 \neq a = a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \in L(e_0, e_{A_1}, \dots, e_{A_m}, e_{A_{m+1}})$ .

Let  $a^{-1} = \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{m+1} a_i \bar{e}_{A_i} \right)$ . Then

$$\begin{aligned}
a \cdot a^{-1} &= \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \right) \left( a_0 e_0 + \sum_{i=1}^{m+1} a_j \bar{e}_{A_j} \right) \\
&= \frac{1}{|a|^2} \left[ a_0^2 e_0 + a_0 \left( \sum_{i=1}^{m+1} a_i e_{A_i} + \sum_{j=1}^{m+1} a_j \bar{e}_{A_j} \right) + \sum_{i=1}^{m+1} a_i^2 e_{A_i} \bar{e}_{A_i} \right. \\
&\quad \left. + \sum_{i < j} a_i a_j (e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i}) \right] = \frac{1}{|a|^2} \left( \sum_{i=0}^{m+1} a_i^2 \right) e_0 = e_0.
\end{aligned}$$

Similarly, one can check the equality  $a^{-1} \cdot a = e_0$ .

Now we prove the necessity. By Lemma 2 we have  $e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i} = 0$  for  $i \neq j$ ;  $i, j \in \{0, 1, \dots, m\}$  and by Lemma 3 we get that  $s = 1$  and

$$\text{either } e_{A_{m+1}} = e_{A_1} e_{A_2} \dots e_{A_m} \text{ or } e_{A_{m+1}} = -e_{A_1} e_{A_2} \dots e_{A_m}.$$

We shall prove that  $m \equiv 2 \pmod{4}$ . From  $e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i} = 0$  for  $i \neq j$ ;  $i, j \in \{0, 1, \dots, m\}$  one gets

$e_{A_i} + \bar{e}_{A_i} = 0$ ,  $i \in \{1, 2, \dots, m\}$ . Hence either  $\#A_i = 4p_i + 1$  or  $\#A_i = 4p_i + 2$  ( $p_i \in \mathbb{N}$ ),  $i \in \{1, 2, \dots, m\}$ . So  $e_{A_i} e_{A_i} = -e_0$  ( $i = 1, 2, \dots, m$ ).

Let  $m \equiv 0 \pmod{4}$  or  $m \equiv 3 \pmod{4}$ . Choosing  $a = e_0 + e_{A_{m+1}}$  and  $b = e_0 - e_{A_{m-1}}$  we find

$$\begin{aligned} ab &= e_0 + e_{A_{m+1}} - e_{A_{m+1}} - e_{A_{m+1}} e_{A_{m+1}} = e_0 - e_{A_1} \dots e_{A_m} \cdot e_{A_1} \dots e_{A_m} \\ &= e_0 - [(-1)^m (-1)^{\frac{m(m-1)}{2}} e_0] = e_0 - (-1)^{\frac{m(m+1)}{2}} e_0 = e_0 - e_0 = 0. \end{aligned}$$

Hence the non-zero numbers  $a$  and  $b$  are not invertible.

Let  $m \equiv 1 \pmod{4}$ . Choosing  $a = e_{A_1} + e_{A_{m+1}}$  and  $b = e_{A_1} - e_{A_{m+1}}$  we get

$$\begin{aligned} ab &= (e_{A_1} + e_{A_{m+1}})(e_{A_1} - e_{A_{m+1}}) \\ &= e_{A_1} e_{A_1} - e_{A_1} e_{A_{m+1}} + e_{A_{m+1}} e_{A_1} - e_{A_{m+1}} e_{A_{m+1}} \\ &= -e_0 - e_{A_1} e_{A_1} e_{A_2} \dots e_{A_m} + e_{A_1} e_{A_2} \dots e_{A_m} e_{A_1} - (-1)^{\frac{m(m+1)}{2}} e_0 \\ &= e_{A_2} \dots e_{A_m} + (-1)^{m-1} e_{A_1} e_{A_1} e_{A_2} \dots e_{A_m} = e_{A_2} \dots e_{A_m} - e_{A_2} \dots e_{A_m} = 0. \end{aligned}$$

Hence  $a$  and  $b$  are not invertible. So  $m \equiv 2 \pmod{4}$ . The theorem is proved.

## References

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