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ON VERY POROSITY AND SPACES OF GENERALIZED UNIFORMLY DISTRIBUTED SEQUENCES

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Abstract. In the paper the porosity structure of sets of generalized uniformly distributed sequences is investigated in the Baire's space.

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1. Introduction and definitions

In [4] the concept of uniformly distributed sequences of positive integers mod m ($m \geq 2$) and uniformly distributed sequences of positive integers in $\mathbf{Z}$ is introduced (see also [1], p. 305).

We recall the notion of Baire's space S of all sequences of positive integers. This means the metric space S endowed with the metric d defined on $S \times S$ in the following way.

Let $x = (x_n)_1^\infty \in S$, $y = (y_n)_1^\infty \in S$. If $x = y$, then $d(x, y) = 0$ and if $x \neq y$, then

$$d(x, y) = \frac{1}{\min\{n : \ x_n \neq y_n\}}.$$

In [2] is proved that the set of all uniformly distributed sequences of positive integers is a set of the first Baire category in (S, d). In the present paper we shall generalize this result to the space of all real sequences.

Denote by (s, d) the metric space of all sequences of real numbers with d Baire's metric.

In the sequel we use the following well-known result of H. Weyl:

Theorem A. *The sequence $x = (x_n)_1^\infty \in s$ is uniformly distributed* (mod 1) *if and only if for each integer $h \neq 0$ the equality*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

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holds (cf. [3], p. 7).

Denote

$$\mathcal{U} = \{x = (x_n)_1^\infty \in s;\ (x_n)_1^\infty \text{ is u. d. mod } 1\},$$

hence from Theorem A we have

$$\mathcal{U} = \left\{ x = (x_n)_1^\infty \in s;\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \ \text{ for each integers } \ h \neq 0 \right\}.$$

We now give definitions and notation from the theory of porosity of sets (cf. [5]-[7]). Let (Y, ϱ) be a metric space. If $y \in Y$ and $r > 0$, then denote by $B(y, r)$ the ball with center y and radius r, i.e.

$$B(y, r) = \{x \in Y : \varrho(x, y) < r\}.$$

Let $M \subseteq Y$. Put

$$\gamma(y, r, M) = \sup\{t > 0 : \quad \exists_{z\in Y} \quad [B(z, t) \subseteq B(y, r)] \wedge [B(z, t) \cap M = \emptyset]\}.$$

Define the numbers:

$$\bar{p}(y, M) = \lim_{r\to 0_+} \sup \frac{\gamma(y, r, M)}{r}, \ \ \underline{p}(y, M) = \lim_{r\to 0_+} \inf \frac{\gamma(y, r, M)}{r}.$$

Obviously the numbers $\bar{p}(y, M)$, $\underline{p}(y, M)$ belong to the interval $[0, 1]$.

A set $M \subseteq Y$ is said to be porous (c-porous) at $y \in Y$ provided that $\bar{p}(y, M) > 0$ ($\bar{p}(y, M) \geq c > 0$). A set $M \subseteq Y$ is said to be σ-porous (σ-c-porous) at $y \in Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets M_n ($n = 1, 2, \ldots$) is porous (c-porous) at y.

Let $Y_0 \subseteq Y$. A set $M \subseteq Y$ is said to be porous, c-porous, σ-porous and σ-c-porous in Y_0 if it is porous, c-porous, σ-porous and σ-c-porous at each point $y \in Y_0$, respectively.

If M is c-porous and σ-c-porous at y, then it is porous and σ-porous at y, respectively.

Every set $M \subseteq Y$ which is porous in Y is non-dense in Y. Therefore every set $M \subseteq Y$ which is σ-porous in Y, is a set of the first category in Y. The converse is not true even in $\mathbf{R}$ (cf. [6]).

A set $M \subseteq Y$ is said to be very porous at $y \in Y$ if $\underline{p}(y, M) > 0$ and very strongly porous at $y \in Y$ if $\underline{p}(y, M) = 1$ (cf. [7] p. 327). A set M is said to be very (strongly) porous in $Y_0 \subseteq Y$ if it is very (strongly) porous at each $y \in Y$.

Obviously, if M is very porous at y, it is porous at y, as well. Analogously, if M is very strongly porous at y, it is 1-porous at y.

Further, a set $M \subseteq Y$ is said to be uniformly very porous in $Y_0 \subseteq Y$ provided that there is a $c > 0$ such that for each $y \in Y_0$ we have $\underline{p}(y, M) \geq c$ (cf. [7], p. 327). In agreement with the previous terminology and in analogy with the notion of σ–porosity, we introduce the following notions. A set $M \subseteq Y$ is said to be uniformly σ–very porous in $Y_0 \subseteq Y$ provided that $M = \bigcup_{n=1}^{\infty} M_n$ and there is a $c > 0$ such that for each $y \in Y_0$ and each $n = 1, 2, \ldots$ we have $\underline{p}(y, M_n) \geq c$.

2. Main Result

In this part of the paper we shall study the set of all uniformly distributed (mod 1) sequences in the space (s, d).

Evidently for an integer $h > 0$ we have

$$\mathcal{U} \subset S^{(h)} = \left\{ x = (x_n)_1^{\infty} \in s; \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \right\} \subseteq \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} F(k, n)$$

for every $k = 1, 2, \ldots$, where

$$F(k, n) = \left\{ x = (x_n)_1^{\infty} \in s; \left| \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h x_j} \right| \leq \frac{1}{k} \right\}.$$

Denote

$$F^*(k, r) = \bigcap_{n=r}^{\infty} F(k, n) \text{ for } k = 1, 2, \ldots, r = 1, 2, \ldots$$

First, for $f : \mathbf{R} \to \mathbf{R}$ let us denote

$$S^{(h)}(f) = \left\{ x = (x_n)_1^{\infty} \in s; \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h f(x_n)} = 0 \right\}$$

and similarly

$$\mathcal{U}(f) = \left\{ x = (x_n)_1^{\infty} \in s; \ (f(x_n))_1^{\infty} \text{ is u. d. mod } 1 \right\}.$$

The next theorem implies, that the set $S^{(h)}$ is σ–very porous in (s, d). (Hence, it follows that σ–very porous in $\mathcal{U}$ too, see Corollary 2.)

Theorem. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Then the set $S^{(h)}(f)$ is uniformly σ–very porous in (s, d).*

Proof. For $f : \mathbf{R} \to \mathbf{R}$ and $k = 1, 2, \dots$ denote by

$$F(f,k,n) = \left\{ x = (x_n)_1^\infty \in s; \quad \left| \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h f(x_j)} \right| \le \frac{1}{k} \right\}.$$

Then we have

$$S^{(h)}(f) \subset \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} F(f,k,n).$$

Let

$$F^*(f,k,r) = \bigcap_{n=r}^{\infty} F(f,k,n).$$

Choose $r \in \mathbf{N}$ fixed. Let $\varepsilon > 0$ and $x \in s$. Further let $\delta > 0$ be such that $\delta < \frac{1}{r}$. Then there exists a positive integer l such that $\frac{1}{l} \le \delta < \frac{1}{l-1}$, (consequently $l > r$). Obviously $S^{(h)}(f) \subseteq \bigcup_{r=1}^{\infty} F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)$. Therefore it suffices to prove

$$\underline{p}\left(x, F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)\right) \ge \frac{1}{2}.$$

Choose a sequence $y \in s$ as follows:

$$y_j = \begin{cases} x_j, & \text{for } j = 1, 2, \dots, l, \\ b, & \text{for } j > l, \end{cases}$$

where b is constant. Evidently $y \in B\left(x, \frac{1}{l}\right)$ and $B\left(y, \frac{1}{[(2+\varepsilon)l]+1}\right) \subset B\left(x, \frac{1}{l}\right)$. We will show

$$B\left(y, \frac{1}{[(2+\varepsilon)l]+1}\right) \cap F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right) = \emptyset.$$

Let $z \in B\left(y, \frac{1}{[(2+\varepsilon)l]+1}\right)$. Then we have

$$\left| \frac{1}{[(2+\varepsilon)l]+1} \sum_{j=1}^{[(2+\varepsilon)l]+1} e^{2\pi i h f(z_j)} \right| \ge \left| \frac{1}{[(2+\varepsilon)l]+1} \sum_{j=l+1}^{[(2+\varepsilon)l]+1} e^{2\pi i h f(z_j)} \right|$$

$$- \left| \frac{1}{[(2+\varepsilon)l]+1} \sum_{j=1}^{l} e^{2\pi i h f(z_j)} \right| \ge \frac{[(2+\varepsilon)l]+1-l}{[(2+\varepsilon)l]+1} - \frac{1}{[(2+\varepsilon)l]+1}$$

$$> \frac{(2+\varepsilon)l - 2l}{[(2+\varepsilon)l]+1} \ge \frac{\varepsilon l}{(2+\varepsilon)l+1} = \frac{\varepsilon}{2+\varepsilon+\frac{1}{l}} > \frac{\varepsilon}{\varepsilon+3} = \frac{1}{\frac{3}{\varepsilon}+1} > \frac{1}{\left[\frac{3}{\varepsilon}\right]+2},$$

thus $z \notin F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r)$. Then

$$\frac{\gamma(x, \delta, F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r))}{\delta} \geq \frac{\frac{1}{[(2+\varepsilon)l]+1}}{\frac{1}{l-1}} \geq \frac{l-1}{(2+\varepsilon)l+1},$$

(i.e.)

$$\underline{p}\left(x, F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)\right) \geq \frac{1}{2+\varepsilon}$$

and letting $\varepsilon \to 0$ we obtain the required inequality.

Remark. Since the set $F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r)$ is closed in s, for each $x \in s \backslash F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)$ holds

$$p\left(x, F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)\right) = 1.$$

Corollary 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Then the set $U(f)$ is uniformly σ–very porous in (s, d).*

Corollary 2. *The set $S^{(h)}$ is uniformly σ–very porous in (s, d) for every h positive integers.*

Proof. It follows from the fact that the function $f(x) = x$, $x \in \mathbf{R}$.

References

[1] KUIPERS – NIEDERREITER, H., Uniform distribution of sequences, Wiley, New York, (1974).

[2] LÁSZLÓ, V. – ŠALÁT, T., Uniformly distributed sequences of positive integers in Baire's space, *Math. Slovaca* **41. no 3** (1991), 277 – 281.

[3] LÁSZLÓ, V. – ŠALÁT, T., The structure of some sequences spaces, and uniform distribution (mod 1), *Periodica Math. Hung.*, **Vol. 10(1)** (1979), 89 – 98.

[4] NIVEN, I., Uniform distribution of sequences of integers, *Trans. Amer. Math. Soc.*, **98** (1961), 52 – 61.

[5] TKADLEC, J., Construction of some non–σ–porous sets of real line, *Real Anal. Exch.*, **9** (1983 – 84), 473 – 482.

[6] ZAJÍČEK, L., Sets of σ–porosity and sets of σ–porosity (q), *Čas. pěst. mat.*, **101** (1976), 350 – 359.

[7] Zajíček, L., Porosity and σ–porosity, *Real Anal. Exch.*, **13** (1987 – 88), 314 – 350.

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