Acta Acad. Paed. Agriensis, Sectio Mathematicae **28** *(2001) 43–53*

ON THE STABILITY OF A SUM FORM FUNCTIONAL EQUATION OF MULTIPLICATIVE TYPE

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Abstract. The stability of a so-called sum form functional equation arising in information theory is proved under certain conditions.

1. Introduction

A function a is additive, a function $M : [0,1] \to \mathbf{R}$ is multilpicative, and a function $l : [0,1] \to \mathbf{R}$ is logarithmic if $a(x+y) = a(x) + a(y)$ for all $x, y \in \mathbf{R}$, $M(xy) = M(x)M(y)$ for all $x, y \in]0,1[$, $M(0) = 0$, $M(1) = 1$, and $l(xy) = l(x)+l(y)$ for all $x, y \in]0,1]$, $l(0) = 0$, respectively.

We define the following sets of complete probability distributions

$$\Gamma_n = \left\{(p_1, \dots, p_n) \in [0,1]^n : \sum_{i=1}^{n} p_i = 1\right\}$$

and

$$\Gamma_n^0 = \left\{(p_1, \dots, p_n) \in]0,1[^n : \sum_{i=1}^{n} p_i = 1\right\}$$

Through the paper I and Δ_n shall denote $[0,1]$ or $]0,1[$ and Γ_n or Γ_n^0, respectively.

Let $n \geq 3$ and $m \geq 3$ be fixed integers, $M_1, M_2 : I \to \mathbf{R}$ be fixed multiplicative functions and $f : I \to \mathbf{R}$ be an unknown function. The functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} f(q_j) + \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} M_2(q_j) \tag{1.1}$$

which holds for all $(p_1, \dots, p_n) \in \Delta_n$ and $(q_1, \dots, q_m) \in \Delta_m$ plays important role in the characterization of information measures.

The general solution of (1.1) is known when M_1 or M_2 is different from the identity function. The $M_1(x) = M_2(x) = x$, $x \in I$ case will be excluded from our investigations, too. In the closed domain case, when the multiplicative functions are power functions, the general solution was given by L. Losonczi and Gy. Maksa in [8].

Theorem 1. *Let $n \geq 3$ and $m \geq 3$ be fixed integers, $\alpha, \beta \in \mathbf{R}$, $\alpha \neq 1$ or $\beta \neq 1$, $M_1(p) = p^\alpha$, $M_2(p) = p^\beta$, $p \in [0,1]$, $0^\alpha = 0^\beta = 0$. The general solution of equation (1.1) is*

$$f(p) = a_1(p) + C(p^\alpha - q^\beta), \quad p \in [0,1] \quad \text{if} \quad \alpha \neq \beta$$
$$f(p) = a_2(p) + p^\alpha l(p), \quad p \in [0,1] \quad \text{if} \quad \alpha = \beta \neq 1$$

where a_1 and a_2 are additive functions, $a_1(1) = a_2(1) = 0$, l is a logarithmic function, and $c \in \mathbf{R}$.

In the open domain case the general solution of (1.1) was given by B. R. Ebanks, P. Kannappan, P. K. Sahoo, and W. Sander in [2]:

Theorem 2. *Let $n \geq 3$ and $m \geq 3$ be fixed integers, $M_1, M_2 :]0,1[\to \mathbf{R}$ be fixed multiplicative functions, M_1 or M_2 is different from the identity function. The general solution of equation (1.1) is*

$$f(p) = a_1(p) + C(M_1(p) - M_2(p)), \quad p \in]0,1[\quad \text{if} \quad M_1 \neq M_2$$
$$f(p) = a_2(p) + M_1(p)l(p) - b, \quad p \in]0,1[\quad \text{if} \quad M_1 = M_2$$

where a_1 and a_2 are additive functions, $a_1(1) = 0$, l is a logarithmic function, $c \in \mathbf{R}$, and

$$b = a_2(1) = 0, \quad \text{if} \quad M_1 = M_2 \notin \{0,1\},$$
$$b = \frac{a_2(1)}{nm}, \quad \text{if} \quad M_1 = M_2 = 0,$$
$$b = \frac{a_2(1)}{nm}(n + m - 1), \quad \text{if} \quad M_1 = M_2 = 1.$$

Applying the methods used in the proof of Theorem 1 in Losonczi-Maksa [8] with arbitrary multiplicative functions (which are not both identity functions) instead of power functions we have the following generalization of Theorem 1.

Theorem 3. *Let $n \geq 3$ and $m \geq 3$ be fixed integers, $M_1, M_2 : [0,1] \to \mathbf{R}$ be fixed multiplicative functions, M_1 or M_2 is different from the identity function. Then the general solution of equation (1.1) is*

$$f(p) = a_1(p) + C(M_1(p) - M_2(p)), \quad p \in [0,1] \quad \text{if} \quad M_1 \neq M_2$$
$$f(p) = a_2(p) + M_1(p)l(p), \quad p \in [0,1] \quad \text{if} \quad M_1 = M_2$$

where a_1 and a_2 are additive functions, $a_1(1) = a_2(1) = 0$, l is a logarithmic function and $c \in \mathbf{R}$.

For the problem of the stability of functional equations in Hyers-Ulam sense we refer to the survey paper of Hyers and Rassias [4]. By the stability problem for equation (1.1) we mean the following: Let $n \geq 3$ and $m \geq 3$ be fixed integers, $M_1, M_2 : I \to \mathbf{R}$ be fixed multiplicative functions, and $0 \leq \varepsilon \in \mathbf{R}$ be fixed. Prove or disprove that the functions $f : I \to \mathbf{R}$ satisfying the funtional inequality

$$(1.2) \qquad \left| \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} f(q_j) - \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} M_i(q_j) \right| \leq \varepsilon$$

for all $(p_1, \ldots, p_n) \in \Delta_n$ and $(q_1, \ldots, q_m) \in \Delta_m$ are the sum of a solution of (1.1) and a bounded function.

The stability of equation (1.1) on closed domain, when the multiplicative functions are power functions was proved in Kocsis-Maksa [6].

Theorem 4. *Let $n \geq 3$ and $m \geq 3$ be fixed integers, $\varepsilon, \alpha, \beta \in R$, $\varepsilon \geq 0$, $\alpha \neq 1$ or $\beta \neq 1$. If the function $f : [0,1] \to \mathbf{R}$ satisfies the inequality (1.2) for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$ then there exists an additive function a, a logarithmic function $l : [0,1] \to \mathbf{R}$, a bounded function $B : [0,1] \to \mathbf{R}$, and $C \in \mathbf{R}$ such that $a(1) = 0$ and*

$$f(p) = a(p) + C(p^{\alpha} - q^{\beta}) + B(p), \quad p \in [0,1] \quad \text{if} \quad \alpha \neq \beta,$$
$$f(p) = a(p) + p^{\alpha} l(p) + B(p), \qquad p \in [0,1] \quad \text{if} \quad \alpha = \beta \neq 1.$$

In this paper we deal with the stability of (1.1) on closed domain and on open domain when the functions M_1 and M_2 are arbitrary multiplicative functions (M_1 or M_2 is different from the identity function).

We notice that the condition $n = m$ or $(n \neq m)$ is essential in the open domain case when zero probabilities are excluded, while it is not essential in the closed domain case.

The basic tool for the proof of the stability theorems is the stability of the sum form functional eqation

$$(1.3) \qquad \sum_{i=1}^{n} \varphi(p_i) = 0,$$

where $n \geq 3$ is a fixed integer, $\varphi : I \to \mathbf{R}$ is an unknown function and (1.3) holds for all $(p_1, \ldots, p_n) \in \Delta_n$. The general solution of equation (1.3) in the closed domain case was given by L. Losonczi and Gy. Maksa in [8] and in the open domain case by L. Losonczi in [7]. In both cases the general solution of (1.3) is

$$(1.4) \qquad \varphi(p) = a(p) - \frac{a(1)}{n}, \quad p \in I,$$

where a is an additive function.

The stability problem for equation (1.3) was solved by Gy. Maksa in [10] on closed domain and by I. Kocsis in [5] on open domain.

Lemma 1. (Maksa [10]) *Let $n \geq 3$ be a fixed integer and $0 \leq \varepsilon \in \mathbf{R}$ be fixed. If the function $\varphi : [0,1] \to \mathbf{R}$ satisfies the inequality*

$$\left| \sum_{i=1}^{n} \varphi(p_i) \right| \leq \varepsilon, \tag{1.5}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, then there exist an additive function A and a bounded function $B : [0,1] \to \mathbf{R}$ such that $B(0) = 0$, $|B(p)| \leq 18\varepsilon$, and

$$\varphi(p) - \varphi(0) = A(p) + B(p), \qquad p \in [0,1].$$

Lemma 2. (Kocsis [5]) *Let $n \geq 3$ be a fixed integer and $0 \leq \varepsilon \in \mathbf{R}$ be fixed. If the function $\varphi :]0,1[\to \mathbf{R}$ satisfies (1.5) for all $(p_1, \ldots, p_n) \in \Gamma_n^0$, then there exist an additive function A and a bounded function $B : [0,1] \to \mathbf{R}$ such that $|B(p)| \leq 220\varepsilon$, and*

$$\varphi(p) = A(p) - \frac{A(1)}{n} + B(p), \qquad p \in]0,1[\,.$$

In what follows the following two lemmata will also be needed.

Lemma 3. *Let A is an additive function, $M : I \to \mathbf{R}$ is a multiplicative function $B : I \to \mathbf{R}$ is a bounded function, and $c \in \mathbf{R}$.*

If $A(x) = M(x) + c$ for all $x \in I$ then

$$A(x) = dx, x \in \mathbf{R}$$

for some $d \in \mathbf{R}$ and

$$M(x) = 0 \quad or \quad M(x) = x, \; x \in I.$$

If $A(x) = M(x) + B(x)$ for all $x \in I$ then

$$A(x) = dx, x \in \mathbf{R}$$

for some $d \in \mathbf{R}$ and

$$M(x) = 0 \quad or \quad M(x) = x^{\alpha}, \; x \in I$$

for some $0 \leq \alpha \in \mathbf{R}$.

Proof. If $A(x) = M(x) + c$ for all $x \in I$ then, because of $A(x) = M(\sqrt{x})^2 + c \geq c$, A is bounded below on I. Thus $A(x) = dx$, $x \in \mathbf{R}$ for some $d \in \mathbf{R}$. (See Aczél [1].) Therefore $M(x) = dx - c$, $x \in I$. Since M is multiplicative we have that $c = 0$ and $d \in \{0, 1\}$.

If $A(x) = M(x) + B(x)$ for all $x \in I$ we have similarly that $A(x) = dx$, $x \in \mathbf{R}$ for some $d \in \mathbf{R}$ and $M(x) = dx - B(x)$, $x \in I$. Thus M is bounded on I that is $M(x) = 0$ or $M(x) = x^\alpha, x \in I$, for some $0 \leq \alpha \in \mathbf{R}$.

Lemma 4. *Let $M_1, M_2 : I \to \mathbf{R}$ be fixed multiplicative functions, $M_1 \neq M_2$, A be an additive function and $c \in \mathbf{R}$. If $M_1(x) - M_2(x) = A(x) + c$ holds for all $x \in I$ then M_1 and M_2 are zero or identity functions of I.*

Proof. Let $a \in I, M_1(a) \neq M_2(a)$. Then from the equations

$$M_1(x) - M_2(x) = A(x) + c \tag{1.6}$$

and

$$M_1(a)M_1(x) - M_2(a)M_2(x) = A(ax) + c$$

we get that

$$M_2(x) = \frac{1}{M_1(a) - M_2(a)}A(ax) - \frac{M_1(a)}{M_1(a) - M_2(a)}A(x) + \frac{c(1 - M_1(a))}{M_1(a) - M_2(a)},$$

that is, there exist an additive function A^* and a constant $c^* \in \mathbf{R}$ such that $M_2(x) = A^*(x) + c^*$ for all $x \in I$. Thus, by Lemma 3, M_2 is zero or identity function of I. Furthermore, by (1.6), we have the same for M_2.

2. The main results

We present two generalizations of Theorem 4. The following theorem says that the functional equation (1.1) is stable on the closed domain.

Theorem 5. *Let $n \geq 3$ and $m \geq 3$ be fixed integers, $0 \leq \varepsilon \in \mathbf{R}$ be fixed and $M_1, M_2 : [0, 1] \to \mathbf{R}$ be fixed multiplicative functions, M_1 or M_2 is different from the identity function. If the function $f : [0, 1] \to \mathbf{R}$ satisfies the inequality (1.2) for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$ then there exists an additive function a, a logarithmic function $l : [0, 1] \to \mathbf{R}$, a bounded function $B : [0, 1] \to \mathbf{R}$, and $C \in \mathbf{R}$ such that*

$$f(p) = a(p) + C(M_1(p) - M_2(p)) + B(p), \quad p \in [0,1] \quad \textit{if} \quad M_1 \neq M_2,$$
$$f(p) = a_2(p) + M_1(p)l(p) + B(p), \quad p \in [0,1] \quad \textit{if} \quad M_1 = M_2.$$

The following theorem states that the functional equation (1.1) is stable on the open domain when $n = m \geq 3$ and $M_1 \neq M_2$.

Theorem 6. *Let $n = m \geq 3$ be a fixed integer, $0 \leq \varepsilon \in \mathbf{R}$ be fixed and $M_1, M_2 :]0,1[\to \mathbf{R}$ be fixed multiplicative functions, $M_1 \neq M_2$. If the function $f :]0,1[\to \mathbf{R}$ satisfies the inequality (1.2) for all $(p_1, \dots, p_m)$, $(q_1, \dots, q_m) \in \Gamma_m^0$ then there exists an additive function a, a bounded function $B :]0,1[\to \mathbf{R}$, and $C \in \mathbf{R}$ such that*

$$f(p) = a(p) + C(M_1(p) - M_2(p)) + B(p), \quad p \in]0,1[\,.$$

The proofs of Theorem 5 and Theorem 6 are based on the following arguments. In the closed and open domain case we use Lemma 1 and Lemma 2, respectively.

Applying Lemma 1 or Lemma 2 for the function

$$\varphi(p, Q) = \sum_{j=1}^{m} (f(pq_j) - M_1(p)f(q_j) - f(p)M_2(q_j))$$

with fixed $Q = (q_1, \dots, q_m) \in \Delta_m$ (1.2) implies that

$$\begin{aligned} \sum_{j=1}^{m} (f(pq_j) - M_1(p) \sum_{j=1}^{m} f(q_j) - f(p) \sum_{j=1}^{m} M_2(q_j)) \\ = A_1(p, Q) + b_1(p, Q) + L_1(Q) \end{aligned} \tag{2.1}$$

holds for all $p \in I$, where $A_1 : \mathbf{R} \times \Delta_m \to \mathbf{R}$ is additive in its first variable and $b_1 : \mathbf{R} \times \Delta_m \to \mathbf{R}$ is bounded. In the closed domain case $L_1(Q) = mf(0) - f(0)\sum_{j=1}^{m} M_2(q_j)$ particulary. Let $P = (p_1, \dots, p_m) \in \Delta_m, p \in I$, write pp_i instead of p in (2.1), $i = 1 \dots m$ and add up the equations we obtained. Thus we get

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{m} (f(pp_i q_j) - M_1(p) \sum_{i=1}^{m} M_1(p_i) \sum_{j=1}^{m} f(q_j) \\ - \sum_{i=1}^{m} f(pp_i) \sum_{j=1}^{m} M_2(q_j)) = A_1(p, Q) + \sum_{i=1}^{m} b_1(pp_i, Q) + mL_1(Q). \end{aligned} \tag{2.2}$$

Write now P instead of Q in (2.1) to obtain

$$\sum_{i=1}^{m} f(pp_i) - M_1(p) \sum_{i=1}^{m} f(p_i) - f(p) \sum_{i=1}^{m} M_2(p_i)) = A_1(p, P) + b_1(p, P) + L_1(P),$$

that is,

$$\sum_{i=1}^{m} f(pp_i) = M_1(p)\sum_{i=1}^{m} f(p_i) - f(p)\sum_{i=1}^{m} M_2(p_i)) + A_1(p,P) + b_1(p,P) + L_1(P).$$

Putting this into (2.2) and collecting the terms symmetric in P and Q on the left hand side we get

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m} f(pp_iq_j) &- f(p)\sum_{i=1}^{m} M_2(p_i)\sum_{j=1}^{m} M_2(q_j)\\ &= M_1(p)\left[\sum_{i=1}^{m} M_1(p_i)\sum_{j=1}^{m} f(q_j) + \sum_{i=1}^{m} f(p_i)\sum_{j=1}^{m} M_2(q_j)\right]\\ &+ A_1(p,P)\sum_{j=1}^{m} M_2(q_j)) + b_1(p)\sum_{j=1}^{m} M_2(q_j)) + L_1(p)\sum_{j=1}^{m} M_2(q_j))\\ &+ A_1(p,Q) + \sum_{i=1}^{m} b_1(pp_i,Q) + mL_1(Q).\end{aligned}$$

Since the right hand side is also simmetric in P and Q we have

$$\begin{aligned}A_1(p,P)&\left[\sum_{j=1}^{m} M_2(q_j) - 1\right] - A_1(p,Q)\left[\sum_{i=1}^{m} M_2(p_i) - 1\right]\\ &= M_1(p)\left[\sum_{j=1}^{m} M_1(q_j)\sum_{i=1}^{m} f(p_i) + \sum_{i=1}^{m} M_2(p_i)\sum_{j=1}^{m} f(q_j)\right.\\ &\left.- \sum_{i=1}^{m} M_1(p_i)\sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} M_2(q_j)\sum_{i=1}^{m} f(p_i)\right]\\ &- L_1(Q)\sum_{i=1}^{m} M_2(p_i) - L_1(P)\sum_{j=1}^{m} M_2(q_j)\\ &+ b_1(p,Q)\sum_{i=1}^{m} M_2(p_i) - b_1(p,P)\sum_{i=j}^{m} M_2(q_j)\\ &+ \sum_{j=1}^{m} b_1(pq_j,P) - \sum_{i=1}^{m} b_1(pp_i,Q) + mL_1(P) - mL_1(Q).\end{aligned} \tag{2.3}$$

The left hand side of (2.3) is aditive in p, while the right hand side can be written in the form $M_1(p)F_1(P,Q)+F_2(p,P,Q)+F_3(P,Q)$, where F_2 is bounded. Applying Lemma 3 with fixed $P,Q\in\Delta_m$ we get that

$$\begin{aligned}(2.4)\qquad & A_1(p,P)\left[\sum_{j=1}^{m}M_2(q_j)-1\right]-A_1(p,Q)\left[\sum_{i=1}^{m}M_2(p_i)-1\right]\\ &=p\left(A_1(1,P)\left[\sum_{j=1}^{m}M_2(q_j)-1\right]-A_1(1,Q)\left[\sum_{i=1}^{m}M_2(p_i)-1\right]\right)\end{aligned}$$

futhermore M_1 is a bounded multiplicative function or

$$(2.5)\qquad \sum_{i=1}^{m}\left(M_1(p_i)-M_2(p_i)\right)\ \sum_{j=1}^{m}f(q_j)=\sum_{j=1}^{m}\left(M_1(q_j)-M_2(q_j)\right)\ \sum_{i=1}^{m}f(p_i).$$

If (2.5) holds then, by (1.4), and by Lemma 4 we have that

$$M_1=M_2\qquad\text{or}$$

$$(2.6)\qquad M_1(p)=p^{\alpha},\ M_2(p)=p^{\beta},\quad p\in I,\ 0\le\alpha\in\mathbf{R},\ 0\le\beta\in\mathbf{R}.$$

Proof of Theorem 5. In the case $M_1\neq M_2$, by (2.6), we can apply Theorem 4.

The case $M_1=M_2$. If the functions M_1 and M_2 are power functions we can apply Theorem 4 again. Suppose now that M_1 is not a power function.

Fix $Q=(q_1,\dots,q_m)\in\Gamma_m$ for which $\sum_{j=1}^{m}M_1(q_j)\neq1$ (exists such a Q) and let

$$(2.7)\qquad a(x)=\frac{A_1(x,Q)-xA_1(1,Q)}{1-\sum_{j=1}^{m}M_1(q_j)},\qquad x\in\mathbf{R}.$$

Then a is additive and $a(1)=0$. From (2.4) we get that

$$(2.8)\qquad A_1(p,P)=pA_1(1,P)+a(p)(1-\sum_{i=1}^{m}M_1(p_i)),$$

while from (2.1), with $p=1$ and $P=Q$, it follows that

$$(2.9)\qquad A_1(1,P)=[f(0)-f(1)]\sum_{i=1}^{m}M_1(p_i)-mf(0)-b_1(1,P),$$

where $p \in [0,1]$, $P \in \Gamma_m$. Equations (2.8) and (2.9) imply that

$$\begin{aligned} A_1(p,P) &= a(p)\left(1 - \sum_{i=1}^{m} M_1(p_i)\right) \\ &+ p\left([f(0) - f(1)]\sum_{i=1}^{m} M_1(p_i) - mf(0) - b_1(1,P)\right) \end{aligned} \tag{2.10}$$

After some calculations we have that

$$\begin{aligned} &\left(b_1(p,P) - p[f(1) + (m-1)f(0) + b_1(1,P)]\right)\sum_{j=1}^{m} M_1(q_j) \\ &= \left(b_1(p,Q) - p[f(1) + (m-1)f(0) + b_1(1,Q)]\right)\sum_{i=1}^{m} M_1(p_i) \\ &+ \sum_{j=1}^{m} b_1(pq_j, P) - \sum_{j=1}^{m} b_1(pp_i, Q) + p[b_1(1,Q) - b_1(1,P)]. \end{aligned} \tag{2.11}$$

Since the right hand side of (2.11) is bounded in Q, while $\sum_{j=1}^{m} M_1(q_j)$ is not, we have

$$b_1(p,P) = p[b_1(1,P) + f(1) + (m-1)f(0)], \quad p \in [0,1],\ P \in \Gamma_m. \tag{2.12}$$

By (2.11), it follows from (2.1) that

$$\begin{aligned} &\sum_{j=1}^{m} \big(h(pq_j) - M_1(p)h(q_j) - h(p)M_1(q_j) + h(0)M_1(q_j) \\ &\quad - p[h(0) - h(1)]M_1(q_j) - h(0) - [h(1) - h(0)]pq_j\big) = 0, \end{aligned}$$

where $h(p) = f(p) - a(p)$, $p \in [0,1]$. Applying Lemma 1 we get that

$$\begin{aligned} &h(pq) - M_1(p)h(q) - M_1(q)h(p) + h(0)M_1(q) - p[h(0) \\ &\quad - h(1)]M_1(q) - h(0) - pq[h(1) - h(0)] + M_1(p)h(0) = A_2(p,q) \end{aligned} \tag{2.13}$$

$p, q \in [0,1]$, where $A_2 : [0,1] \times [0,1] \to \mathbf{R}$ is additive in its second variable. Define the function H on $[0,1]$ by $H(p) = h(p) - h(0)$. Thus (2.13) can be written in the form

$$H(pq) - M_1(p)H(q) - M_1(q)h(p) + H(1)pM(q) = A_2(p,q). \tag{2.14}$$

A calculation shows that the function $G : [0,1]^2 \to \mathbf{R}$ defined by

$$G(p,q) = H(p,q) - M_1(p)h(q) - M_1(q)H(p) \tag{2.15}$$

satisfies the equation

$$G(pq, r) + M(r)G(p, q) = G(p, qr) + M(p)G(q, r), \quad p, q, r \in [0, 1]. \tag{2.16}$$

From (2.14) and (2.15) we have that $G(p, q) = A_2(p, q) - H(1)M(q)$. With (2.16) this implies that

$$A_2(p, qr) - A_2(pq, r) + M_1(p)A_2(q, r) = M_1(r)[A_2(p, q) - H(1)(pq - M_1(p)q)].$$

The left hand side is additive in the variable r and the multiplicative function M_1 is not the identity function so $A_2(p, q) = H(1)(p - M_1(p))q$ thus (2.14) goes over into

$$H(pq) - H(1)pq = M_1(p)(H(q) - H(1)q) + M_1(q)(H(p) - H(1)p), \tag{2.17}$$

where $p, q \in [0, 1]$. Let $l : [0, 1] \to \mathbf{R}$, $l(0) = 0$ and

$$l(p) = \frac{H(p) - H(1)p}{M_1(p)}, \qquad p \in [0, 1].$$

Then (2.17) shows that l is a logarithmic function and for all $p \in [0, 1]$ we have

$$f(p) = a(p) + h(p) = a(p) + H(p) + h(0) = a(p) + M(p)l(p) + H(1)p + h(0).$$

With $B(p) = H(1)p + h(0), p \in [0, 1]$ we obtain the statement of the theorem.

Proof of Theorem 6. Here $n = m, M_1 \neq M_2$ and, by (2.6), M_1 and M_2 are power functions, that is, $M_1(p) = p^\alpha, M_2(p) = p^\beta$, $p \in]0, 1[$ for some $0 \leq \alpha \in \mathbf{R}$, $0 \leq \beta \in \mathbf{R}$. Interchanging P and Q in (1.2) and applying the triangle inequality we have

$$\left| \sum_{j=1}^{m} (q_j^\alpha - q_j^\beta) \sum_{i=1}^{m} f(p_i) - \sum_{i=1}^{m} (p_i^\alpha - p_i^\beta) \sum_{j=1}^{m} f(q_j) \right| \leq 2\varepsilon. \tag{2.18}$$

By Lemma 2 we get

$$f(p) = A(p) + c_1 p^\alpha + c_2 p^\beta + b(p), \qquad p \in]0, 1[,$$

where A is an additive function, $b :]0, 1[\to \mathbf{R}$ is a bounded function, and $c_1, c_2 \in \mathbf{R}$. With the definitions

$$a(p) = A(p) - pA(1), \qquad p \in \mathbf{R}$$

$$B(p) = b(p) + pA(1) + (c_1 + c_2)p^\alpha, \qquad p \in]0, 1[$$

and

$$C = -c_2$$

our theorem is proved.

Remark. It is clear from the paper that some open problem remains connected with the stability of equation (1.1). For example the case $M_1 = M_2$ or $M_1 \neq M_2$ *and* $n \neq m$. The stability problem is essentially solved in the open domain case.

References

[1] Aczél, J., Lectures on Functional Equations and Thier Applications, *Academic Press New York and London*, 1966.

[2] Ebanks, B. R., Kannappan, P. , Sahoo P. K. and Sander W., Characterizations of sum form information measures on open domain, *Aequ. Math.* **54** (1997), 1–30.

[3] Ger R., The singular case in the stability behaviour of linear mappings, *Grazer Math. Ber.* **316** (1991), 59–70.

[4] Hyers, D. H. and Rassias, T. M., Approximate homomorfisms, *Aequ. Math.* **44** (1992), 125–153.

[5] Kocsis, I., Stability of a sum form functional equation on open domain, *Publ. Math. Debrecen* **57** (2000)(1–2), 135–143.

[6] Kocsis, I. and Maksa, Gy., The stability of a sum form functional equation arising in information theory, *Acta Math. Hungar.* **79** (1–2) (1998), 53–62.

[7] Losonczi, L., Functional equation of sum form, *Publ. Math. Debrecen* **32** (1985), 56–71.

[8] Losonczi, L. and Maksa, Gy., The general solution of a functional equation of information theory, —it Glasnik Mat. **16** (36) (1981), 261–266.

[9] Losonczi, L. and Maksa, Gy., On some functional equations of the information theory, *Acta Math. Acad. Sci. Hungar.* **39** (1982), 73–82.

[10] Maksa, Gy., On the stability of a sum form equation, *Results in Mathematics* **26** (1994), 342–347.

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