MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY IV.

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Abstract. It is proved that if an integer-valued completely multiplicative function f with $f(n) \neq 0$ $(\forall n \in \mathbf{N})$ and a polynomial $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in \mathbf{Q}[x]$ satisfy the relation

$$A_P P(E) f(n+m) \equiv A_P P(E) f(n) \pmod{m}$$

for a suitable non-zero integer A_P and for all $n, m \in \mathbf{N}$, where

$$P(E) f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k),$$

then there is a non-negative integer α such that $f(n) = n^\alpha$ for all $n \in \mathbf{N}$. A similar result is true for $P(x) = (x-1)^k$ and a multiplicative function f.

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1. Introduction

An arithmetical function $f (f(n) \not\equiv 0)$ is said to be multiplicative if $(n, m) = 1$ implies

$$f(nm) = f(n) f(m),$$

and it is called completely multiplicative if this equation holds for all positive integers n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of all integer-valued multiplicative and completely multiplicative functions, respectively. Throughout this paper we apply the usual notations, i.e. $\mathcal{P}$ denotes the set of primes, $\mathbf{N}$ the set of positive intgers and $\mathbf{Q}$ the set of rational numbers, respectively.

The problem concerning the characterization of some arithmetical functions by congruence properties was studied by several authors. The first result of this type was found by M. V. Subbarao [9], namely he proved in 1966 that if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(n) \pmod{m} \tag{1}$$

for all $n, m \in \mathrm{N}$, then $f(n)$ is a power of n with non-negative integer exponent. In [4] among others we extended this result by proving that if $f \in \mathcal{M}$ and (1) holds for all $n \in \mathrm{N}$ and for all $m \in \mathcal{P}$, then $f(n)$ also is of the same form. For further results and generalizations of the above problem we refer the papers [1] and [4]-[8].

Let

$$P(x) = a_0 + a_1 x + \cdots + a_k x^k \quad (a_k \neq 0)$$

be an arbitrary polynomial with integer coefficients. In the space of the sequences $\{x_1, x_2, \ldots\}$ let E, I, Δ denote the operators defined by the following relations

$$Ex_n = x_{n+1}, \quad Ix_n = x_n, \quad \Delta x_n = x_{n+1} - x_n.$$

For the polynomial $P(x)$ and the function $f(n)$ we have

$$P(E)f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k).$$

For any fixed subsets A, B of N we shall denote by $\mathcal{K}_P(A, B)$ the set of all $f \in \mathcal{M}$ for which

$$P(E)f(n+m) \equiv P(E)f(n) \pmod{m} \tag{2}$$

holds for all $n \in A$ and $m \in B$. It is obvious that

$$\varphi_a(n) = n^a \tag{3}$$

is a solution of (2) for every non-negative integer a and for every triplet (P, A, B). In the case $P(x) = 1$, for example, from the result of [4], we have

$$\mathcal{K}_P(\mathrm{N}, \mathcal{P}) = \{\varphi_0, \varphi_1, \ldots, \varphi_a, \ldots\}$$

and

$$\mathcal{K}_P(\mathcal{P}, \mathrm{N}) = \{\varphi_0, \varphi_1, \ldots, \varphi_a, \ldots\},$$

where $\varphi_a(n)$ is defined in (3).

We are interested for a characterization of those triplets (P, A, B) for which

$$\mathcal{K}_P(A, B) = \{\varphi_0, \varphi_1, \ldots, \varphi_a, \ldots\} \tag{4}$$

is satisfied. In [5]-[6] we proved that (4) holds for the following two cases:

$$P(x) = (x-1)^k \ \ (k \in \mathrm{N}), \quad A = \mathrm{N}, \quad B = \mathcal{P}, \tag{a}$$

$$(b) \qquad P(x) = x^M - 1 \quad (M \in \mathbf{N}), \quad A = \mathbf{N}, \quad B = \mathcal{P}.$$

Hence we apply the method of I. Kátai [2]-[3] to prove the following.

Theorem 1. *Let $f \in \mathcal{M}^*$ with condition*

$$f(n) \neq 0 \quad \text{for all} \quad n \in \mathbf{N}. \tag{5}$$

Let $P(x)$ be a non-zero polynomial with rational coefficients for which there exists a suitable non-zero integer A_P such that

$$A_P P(E) f(n+m) \equiv A_P P(E) f(n) \pmod{m} \tag{6}$$

for all $n \in \mathbf{N}$ and $m \in \mathbf{N}$. Then there is a non-negative integer α such that

$$f(n) = n^\alpha \quad \text{for all} \quad n \in \mathbf{N}. \tag{7}$$

We mention that in the special case $P(x) = (x-1)^k$, Theorem 1 is true under the assumption $f \in \mathcal{M}$.

Theorem 2. *Let $f \in \mathcal{M}$ and let $A \neq 0$, $k \geq 0$ be integers. If $\Delta^k f(n)$ satisfies the relation*

$$A\Delta^k f(n+m) \equiv A\Delta^k f(n) \pmod{m} \tag{8}$$

for all $n \in \mathbf{N}$ and $m \in \mathbf{N}$, then (7) holds.

2. Proof of Theorem 2

In the proof of Theorem 2 we shall use the following results.

Lemma 1. *Let $f(n)$ be an integer-valued arithmetic function and let $k \in \mathbf{N}$, $Q \in \mathbf{N}$. If $\Delta^k f(n)$ satisfies the relation*

$$\Delta^k f(n+Q) \equiv \Delta^k f(n) \pmod{Q} \tag{9}$$

for all $n \in \mathbf{N}$, then for $s = 1, 2, \ldots, k$

$$\Delta^{k-s} f(n+tQ) - \Delta^{k-s} f(n) \equiv \sum_{j=0}^{s-1} \binom{n-1}{j} \Delta_f^{k-s+j}(Q,t) \pmod{Q} \tag{10}$$

holds for all $n \in \mathrm{N}$, $t \in \mathrm{N}$, *where*

$$\Delta_f^i(Q,t) := \Delta^i f(1+tQ) - \Delta^i f(1) \quad (i = 0, 1, \ldots).$$

Furthermore, if Q *is a prime, then* (9) *implies that*

$$(11) \qquad \Delta_f^{k-s}(Q,t) \equiv \sum_{j=0}^{[\frac{s-1}{Q}]} \binom{t}{j+1} \Delta_f^{k-s+jQ}(Q,1) \pmod{Q}$$

holds for all $t \in \mathrm{N}$, *where* $[x]$ *denotes the largest integer not exceeding* x.

This lemma and its proof can be found in [5] (see Lemma 1-2).

Lemma 2. *Let* $\alpha \in \mathrm{N}$ *and* $f \in \mathcal{M}$. *If*

$$(12) \qquad f(n + p^\alpha) \equiv f(n) \pmod{p}$$

for all $n \in \mathrm{N}$ *and* $p \in \mathcal{P}$, *then* $f \in \mathcal{M}^*$ *and for each* $q \in \mathcal{P}$

$$f(q) = q^{a(q)},$$

where $a(q) \geq 0$ *is an integer.*

This lemma is indentical to Lemma 3 in [5].

Now we prove Theorem 2.

Assume that $f \in \mathcal{M}$ and (8) is true for all $n, m \in \mathrm{N}$. First we shall prove that there exists an $\alpha \in \mathrm{N}$ such that (12) holds for all $n \in \mathrm{N}$ and for all $p \in \mathcal{P}$. If $k = 0$, then (12) is obviously true.

Assume that $k \geq 1$ be an integer. Let α be a fixed positive integer such that

$$(13) \qquad p_0 := \max(|A|, k-1) < 2^{\alpha-1}$$

Since

$$A\Delta^k f(n) = \Delta^k(Af(n)),$$

by (8) it follows that

$$\Delta^k(Af(n + p^{\alpha-1})) \equiv \Delta^k(Af(n)) \pmod{p^{\alpha-1}}$$

holds for all $n \in \mathrm{N}$ and for all $p \in \mathcal{P}$. Thus, by using Lemma 1 and (13), for $s = 1, 2, \ldots, k$ we have

$$(14) \quad \Delta^{k-s} f(n + tp^{\alpha-1}) - \Delta^{k-s} f(n) \equiv \sum_{j=0}^{s-1} \binom{n-1}{j} \Delta_f^{k-s+j}(p^{\alpha-1}, t) \pmod{p}$$

holds for all $n, t \in \mathbf{N}$, $p \in \mathcal{P}$. Applying (14) in the case $n = 1 + ip^{\alpha-1}$ and $t = 1$, one can deduce from (13) that

$$\Delta^{k-s} f(1+(i+1)p^{\alpha-1}) - \Delta^{k-s} f(1+ip^{\alpha-1}) \equiv \sum_{j=0}^{s-1} \binom{ip^{\alpha-1}}{j} \Delta_f^{k-s+j}(p^{\alpha-1}, 1)$$

$$\equiv \Delta_f^{k-s}(p^{\alpha-1}, 1) \pmod{p}, \tag{15}$$

since it is obvious that for a prime p

$$\binom{ip^{\alpha-1}}{j} \equiv 0 \pmod{p} \quad \text{if} \quad 1 \le j < p^{\alpha-1}.$$

From (15) we infer that

$$\Delta_f^{k-s}(p^{\alpha-1}, t) \equiv t\Delta_f^{k-s}(p^{\alpha-1}, 1) \pmod{p},$$

and so

$$\Delta_f^0(p^{\alpha-1}, p) \equiv \Delta_f^1(p^{\alpha-1}, p) \equiv \cdots \equiv \Delta_f^{k-1}(p^{\alpha-1}, p) \equiv 0 \pmod{p} \tag{16}$$

holds for all $p \in \mathcal{P}$. By using (14) with $k = s$ and $t = p$, (16) implies (12). Thus, (12) is proved.

Now, from Lemma 2 we have $f \in \mathcal{M}^*$ and

$$f(q) = q^{a(q)} \tag{17}$$

for each $q \in \mathcal{P}$, where $a(q) \ge 0$ is an integer.

It is clear from (8) that

$$\Delta^k f(n+p) \equiv \Delta^k f(n) \pmod{p}$$

for all $n \in \mathbf{N}$ and $p \in \mathcal{P}$ satisfying the condition $p > |A|$. By using (11) in the case $k = s$, we have

$$f(1+tp) - f(1) \equiv t\,(f(1+p) - f(1)) \pmod{p} \tag{18}$$

for all $t \in \mathbf{N}$ and for every prime $p > p_0 := \max(|A|, k-1)$, because $\left[\frac{k-1}{p}\right] = 0$ for $p \ge k$. Considering $t = p+2$ and taking account (18) we get

$$(f(1+p) - 1)^2 \equiv 0 \pmod{p},$$

and so by (18) we have

$$f(1+tp)-f(1)\equiv 0 \pmod p \tag{19}$$

for all $t\in\mathbf{N}$ and for every prime $p>p_0$.

Let q,r be distinct primes and let $a(q)\ge a(r)$. Then there is a prime p such that

$$p>\max(p_0,q^{a(q)-a(r)}) \quad \text{and} \quad qr^s-1\equiv 0 \pmod p$$

for some positive integer s. Using (19), we have $f(qr^s)\equiv f(1)=1 \pmod p$ and

$$f(qr^s)=q^{a(q)}r^{sa(r)}\equiv q^{a(q)-a(r)} \pmod p,$$

which implies $a(p)=a(q)=\alpha$. Hence, $f(n)=n^\alpha$ for all $n\in\mathbf{N}$. This completes the proof of Theorem 2.

3. Proof of Theorem 1

Let $f\in\mathcal{M}^*$ and $f(n)\ne 0$ for all $n\in\mathbf{N}$. We denote by I_f the set of all polinomials P with rational coefficients for which there exists a suitable non-zero integer A_P such that

$$A_PP(E)f(n+m)\equiv A_PP(E)f(n) \pmod m$$

holds for all $n,m\in\mathbf{N}$. By our assumption (6), we have $I_f\ne\emptyset$. It is clear to check that

(i) $cP(x)\in I_f$ for every $P\in I_f$, $c\in\mathbf{Q}$

(ii) $P(x)+P'(x)\in I_f$ for every P, $P'\in I_f$

(iii) $xP(x)\in I_f$ for every $P\in I_f$. Thus, (i)-(iii) show that I_f is an ideal in $\mathbf{Q}[x]$.

Let

$$S(x)=c_0+c_1x+\cdots+c_kx^k \quad (c_k=1)$$

be a polynomial of minimum degree in I_f. If $k=0$, then Theorem 1 follows from Theorem 2. In the following we assume that $k\ge 1$. Let

$$S(x)=(x-\theta_1)\dots(x-\theta_k).$$

From the fundamental theorem of symmetric polynomials it follows that for a fixed integer $s\ge 1$ the polynomial

$$\prod_{j=1}^{k}\frac{x^s-\theta_j^s}{x-\theta_j}$$

has rational coefficients, consequently

$$Q_s(x^s) = (x^s - \theta_1^s)\dots(x^s - \theta_k^s) \in I_f.$$

Then, by the definition of I_f there is a non-zero integer A_s such that

$$A_s Q_s(E^s) f(n+m) \equiv A_s Q_s(E^s) f(n) \pmod{m} \tag{20}$$

for all $n, m \in \mathbf{N}$. On the other hand, by using the fact $f \in \mathcal{M}^*$, we have

$$Q_s(E^s) f(sn) = f(s) Q_s(E) f(n). \tag{21}$$

Therefore, (20) and (21) imply that

$$A_s Q_s(E^s) f\left[s(n+m)\right] \equiv A_s Q_s(E^s) f(sn) \pmod{sm}$$

and

$$A_s f(s) Q_s(E) f(n+m) \equiv A_s f(s) Q_s(E) f(n) \pmod{sm} \tag{22}$$

for all $n, m \in \mathbf{N}$. Since $f(s) \neq 0$ and $f(s)$ is an integer, (22) shows that $Q_s(x) \in I_f$. Thus

$$\delta(x) = (S(x),\ Q_s(x)) \in I_f$$

and so $\deg \delta(x) = k$, $S(x) = Q_s(x)$. This implies that

$$\{\theta_1, \dots, \theta_k\} = \{\theta_1^s, \dots, \theta_k^s\}$$

for all $s \in \mathbf{N}$, consequently

$$\theta_1 = \dots = \theta_k = 1 \quad \text{and} \quad S(x) = (x-1)^k.$$

Thus, Theorem 1 follows directly from Theorem 2.

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