Annales Mathematicae et Informaticae
47 (2017) pp. 219–225
http://ami.uni-eszterhazy.hu

On bounded and unbounded curves determined by their curvature and torsion

Oleg Zubelevich*

Department of Theoretical mechanics, Mechanics and Mathematics Faculty
M. V. Lomonosov moscow State University, Russia, Moscow
ozubel@yandex.ru

Submitted December 7, 2016 — Accepted January 23, 2017

Abstract

We consider a curve in $\mathbb{R}^3$ and provide sufficient conditions for the curve to be unbounded in terms of its curvature and torsion. We also present sufficient conditions on the curvatures for the curve to be bounded in $\mathbb{R}^4$.

Keywords: Curves, intrinsic equation, curvature, torsion, Frenet-Serret formulas.

MSC: 53A04

1. Introduction

This short note concerns a smooth curve γ in the standard three-dimensional Euclidean space $\mathbb{R}^3$. It is well known that the curve is uniquely defined (up to translations and rotations of $\mathbb{R}^3$) by its curvature $\kappa(s)$ and its torsion $\tau(s)$, the argument s is the arc-length parameter. The pair $(\kappa(s), \tau(s))$ is called the intrinsic equation of the curve.

In the sequel we assume that $\kappa, \tau \in C[0, +\infty)$.

To obtain the radius-vector of the curve γ one must solve the system of Frenet-Serret equations:

$$\begin{aligned} \boldsymbol{v}'(s) &= \kappa(s)\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) &= -\kappa(s)\boldsymbol{v}(s) + \tau(s)\boldsymbol{b}(s), \end{aligned} \tag{1.1}$$

*Partially supported by grant RFBR 15-01-03747.

$$\boldsymbol{b}'(s) = -\tau(s)\boldsymbol{n}(s).$$

The vectors $\boldsymbol{v}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$ stand for the Frenet-Serret frame at the curve's point with parameter s. Then the radius-vector of the curve is computed as follows $\boldsymbol{r}(s) = \int_0^s \boldsymbol{v}(\xi)d\xi + \boldsymbol{r}(0)$.

If the curve γ is flat (it is so iff $\tau(s) = 0$) then the system (1.1) is integrated explicitly. In three dimensional case nobody can integrate this system with arbitrary sooth functions τ, κ.

So we obtain a very natural and pretty problem: to restore the properties of the curve γ having the curvature $\kappa(s)$ and the torsion $\tau(s)$.

For example, under which conditions on the functions κ, τ a curve γ is closed? This is a hard open problem. There may by another question: Which are sufficient conditions for the whole curve to be contained in a sphere? This question is much simpler. Such a type questions have been discussed in [4, 3, 5].

There is a sufficient condition for the curve to be unbounded [1]. In this article the condition is formulated in terms of curvature only and this condition is valid in a big class of spaces, including Hilbert spaces and Riemannian manifolds of non-positive curvature.

In general case, (1.1) is a linear system of ninth order with matrix depending on s. To describe the properties of γ one must study this system.

In this note we formulate and prove some sufficient conditions for unboundedness of the curve γ.

We also present sufficient conditions for the curve to be bounded in the four dimensional Euclidean space.

It is interesting that in $\mathbb{R}^m$ of odd m the curves are in generic case unbounded but for the even m they are generically bounded. Some justification of this very informal observation is given below.

2. Main theorem

We shall say that γ is unbounded iff $\sup_{s\geq 0} |\boldsymbol{r}(s)| = \infty$.

Theorem 2.1. *Suppose there exists a function $\lambda(s)$ such that functions*

$$k(s) = \lambda(s)\kappa(s), \quad t(s) = \lambda(s)\tau(s)$$

are monotone[1] *and belong to $C[0, \infty)$.*

Introduce a function $T(s) = \int_0^s t(\xi)d\xi$.

Suppose also that the following equalities hold

$$\lim_{s\to\infty} T(s) = \infty, \quad \lim_{s\to\infty} \frac{k(s)}{T(s)} = \lim_{s\to\infty} \frac{t(s)}{T(s)} = 0. \tag{2.1}$$

[1] E.g. one of these functions, $k(s)$ is monotonically increasing: $s' < s'' \Rightarrow k(s') \leq k(s'')$, $s', s'' \in [0, \infty)$ while other one $t(s)$ is monotonically decreasing: $s' < s'' \Rightarrow t(s') \geq t(s'')$, $s', s'' \in [0, \infty)$. The inverse situation is also allowed, or the both functions can be increasing or decreasing simultaneously.

Then the curve γ is unbounded.

The proof of this theorem is contained in Section 4.1.

Putting $\lambda = 1/\tau$ in this Theorem , we deduce the following corollary.

Corollary 2.2. *Suppose that the function $\kappa(s)/\tau(s)$ is monotone and*

$$\lim_{s\to\infty} \frac{\kappa(s)}{s\cdot\tau(s)} = 0. \tag{2.2}$$

Then the curve γ is unbounded.

Note that the geodesic curvature of the tantrix[2] $\kappa_T(s)$ is equal to $\tau(s)/\kappa(s)$ [3]. So that formula (2.2) can be rewritten as follows

$$\lim_{s\to\infty} \kappa_T(s)s = \infty.$$

Theorem 2.1 is not reduced to Corollary 2.2. Consider an example. Let the curve γ be given by

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{1+s}.$$

Since $\tau(s) \to 0$ as $s \to \infty$ it may seem that this curve is about a circle with $\kappa(s) = 1$. Nevertheless applying Theorem 2.1 with $\lambda = 1$ we see that the curve γ is unbounded.

Consider a system which consists of (1.1) together with the equation $\boldsymbol{r}'(s) = \boldsymbol{v}(s)$. From viewpoint of stability theory, Theorem 2.1 states that under certain conditions this system is unstable.

Since $|\boldsymbol{r}(s)| = O(s)$ as $s \to \infty$, this instability is too weak to study it by standard methods such as the Lyapunov exponents method.

3. Supplementary remarks: Bounded curves in $\mathbb{R}^4$

Actually the above developed technique can be generalized to the curves in any multidimensional Euclidean space $\mathbb{R}^m$. For the case of the odd m we can prove a theorem similar to Theorem 2.1. But for the case when m is even our method allows to obtain sufficient conditions for the curve to be bounded.

In this section we illustrate such an effect. To avoid of big formulas we consider only the case $m = 4$.

So let a curve $\gamma \subset \mathbb{R}^4$ be given by its curvatures

$$\kappa_i(s) \in C[0,\infty), \quad i = 1, 2, 3.$$

And let $\boldsymbol{v}_j(s)$, $j = 1, 2, 3, 4$ be the Frenet-Serret frame.

[2]The tangential spherical image of the curve γ is the curve on the unit sphere. This curve has the radius-vector $\boldsymbol{r}'(s)$.

Then the Frenet-Serret equations are

$$\frac{d}{ds}\begin{pmatrix}\boldsymbol{v_1}\\ \boldsymbol{v_2}\\ \boldsymbol{v_3}\\ \boldsymbol{v_4}\end{pmatrix}(s) = A(s)\begin{pmatrix}\boldsymbol{v_1}\\ \boldsymbol{v_2}\\ \boldsymbol{v_3}\\ \boldsymbol{v_4}\end{pmatrix}(s),$$

$$A(s) = \begin{pmatrix} 0 & \kappa_1(s) & 0 & 0\\ -\kappa_1(s) & 0 & \kappa_2(s) & 0\\ 0 & -\kappa_2(s) & 0 & \kappa_3(s)\\ 0 & 0 & -\kappa_3(s) & 0\end{pmatrix}$$

Theorem 3.1. *Suppose that the function $\kappa_1(s)\kappa_3(s)$ does not take the value zero. The functions*

$$f_1(s) = \frac{1}{\kappa_1(s)}, \quad f_2(s) = \frac{\kappa_2(s)}{\kappa_1(s)\kappa_3(s)}$$

are monotone and

$$\sup_{s\ge 0}|f_i(s)| < \infty, \quad i = 1, 2.$$

Then the curve γ is bounded.

The proof of this theorem is contained in Section 4.2.

4. Proofs

4.1. Proof of Theorem 2.1

Let us expand the radius-vector by the Frenet-Serret frame

$$\boldsymbol{r}(s) = r_1(s)\boldsymbol{v}(s) + r_2(s)\boldsymbol{n}(s) + r_3(s)\boldsymbol{b}(s).$$

Differentiating this formula we obtain

$$\boldsymbol{v}(s) = r_1'(s)\boldsymbol{v}(s) + r_2'(s)\boldsymbol{n}(s) + r_3'(s)\boldsymbol{b}(s) \\ + r_1(s)\boldsymbol{v}'(s) + r_2(s)\boldsymbol{n}'(s) + r_3(s)\boldsymbol{b}'(s).$$

Using the Frenet-Serret equations, one yields

$$r'(s) = \begin{pmatrix} 0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix} r(s) + \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad r = \begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}. \tag{4.1}$$

The author was informed about system (4.1) by Professor Ya. V. Tatarinov.

Let us multiply both sides of system (4.1) by the row-vector

$$\lambda(s)\big(\tau(s), 0, \kappa(s)\big)$$

from the left:

$$t(s)r_1'(s) + k(s)r_3'(s) = t(s).$$

Then we integrate this equation:

$$\int_0^s t(a)r_1'(a)da + \int_0^s k(a)r_3'(a)da = T(s). \tag{4.2}$$

From the Second Mean Value Theorem [2], we know that there is a parameter $\xi \in [0, s]$ such that

$$\begin{aligned}\int_0^s t(a)r_1'(a)da &= t(0)\int_0^\xi r_1'(a)da + t(s)\int_\xi^s r_1'(a)da \\ &= t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big)\end{aligned}$$

By the same argument for some $\eta \in [0, s]$ we have

$$\int_0^s k(a)r_3'(a)da = k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big).$$

Thus formula (4.2) takes the form

$$\begin{aligned}t(0)\big(r_1(\xi) - r_1(0)\big) &+ t(s)\big(r_1(s) - r_1(\xi)\big) \\ &+ k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big) = T(s).\end{aligned} \tag{4.3}$$

Since the Frenet-Serret frame is orthonormal we have

$$|\boldsymbol{r}(s)|^2 = r_1^2(s) + r_2^2(s) + r_3^2(s) = |r(s)|^2.$$

Assume the Theorem is not true: the curve γ is bounded, i.e. $\sup_{s\geq 0}|\boldsymbol{r}(s)| < \infty$. Then due to conditions (2.1) the left side of formula (4.3) is $o(T(s))$ as $s \to \infty$. This contradiction proves the theorem.

The Theorem is proved.

4.2. Proof of Theorem 3.1

Let $\boldsymbol{r}(s)$ be a radius-vector of the curve γ. Then one can write

$$\boldsymbol{r}(s) = \sum_{i=1}^4 r_i \boldsymbol{v}_i(s), \quad \boldsymbol{r}'(s) = \boldsymbol{v}_1(s).$$

Similarly as in the previous section, due to the Frenet-Serret equations this gives

$$r'(s) = A(s)r(s) + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad r = \begin{pmatrix}r_1\\r_2\\r_3\\r_4\end{pmatrix}.$$

First we multiply this equation by $r'^T(s)A^{-1}(s)$, $\quad(\det A = (\kappa_1\kappa_3)^2)$:

$$r'^T(s)A^{-1}(s)r'(s) = r'^T(s)r(s) + r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{4.4}$$

Since A^{-1} is a skew-symmetric matrix we have $r'^T(s)A^{-1}(s)r'(s) = 0$, and some calculation yields

$$r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

Then formula (4.4) takes the form

$$-\frac{1}{2}\Big(|r(s)|^2\Big)' = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

Integrating this formula we obtain

$$-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) = \int_0^s r_2'(a)f_1(a) + r_4'(a)f_2(a)da.$$

By the same argument which was employed to obtain formula (4.3), it follows that

$$\begin{aligned}&-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) =\\ &f_1(0)\big(r_2(\xi) - r_2(0)\big) + f_1(s)\big(r_2(s) - r_2(\xi)\big)+\\ &f_2(0)\big(r_4(\eta) - r_4(0)\big) + f_2(s)\big(r_4(s) - r_4(\eta)\big),\end{aligned} \tag{4.5}$$

here $\xi, \eta \in [0, s]$.

To proceed with the proof assume that the curve γ be unbounded:

$$\sup_{s\geq 0} |r(s)| = \infty.$$

Take a sequence s_k such that

$$|r(s_k)| = \max_{s\in[0,k]} |r(s)|, \quad k \in \mathbb{N}, \quad s_k \in [0, k].$$

It is easy to see that

$$s_k \to \infty, \quad |r(s)| \leq |r(s_k)|, \quad s \in [0, s_k]$$

and $|r(s_k)| \to \infty$ as $k \to \infty$.

Substitute this sequence to formula (4.5):

$$-\frac{1}{2}\Big(|r(s_k)|^2 - |r(0)|^2\Big) =$$
$$f_1(0)\big(r_2(\xi_k) - r_2(0)\big) + f_1(s_k)\big(r_2(s_k) - r_2(\xi_k)\big) +$$
$$f_2(0)\big(r_4(\eta_k) - r_4(0)\big) + f_2(s_k)\big(r_4(s_k) - r_4(\eta_k)\big), \tag{4.6}$$

here $\xi_k, \eta_k \in [0, s_k]$ and thus $|r_2(\xi_k)| \leq |r(s_k)|, \quad |r_4(\eta_k)| \leq |r(s_k)|$.

Due to conditions of the Theorem and the choice of the sequence s_k the right-hand side of formula (4.6) is $O(|r(s_k)|)$ as $k \to \infty$. But the left-hand one is of order $-|r(s_k)|^2/2$. This gives a contradiction.

The Theorem is proved.

References

[1] S. Alexander, R. Bishop, R. Ghrist, Total curvature and simple pursuit on domains of curvature bounded above, Geometriae Dedicata, 147(2010), 275–290.

[2] R. Courant, Differential and Integral Calculus, vol. 1, John Wiley and Sons, 1988.

[3] W. Frenchel, On the differential geometry of closed space curves, Bull. Amer. Math. Soc., 57(1951), 44–54.

[4] P. W. Gifford, Some refinements in theory of specialized space curves, Amer. Math. Monthly, 60(1953), 384–393.

[5] Yung-Chow Wong, Hon-Fei Lai, A Critical Examination of the Theory of Curves in Three Dimensional Differential Geometry, Tohoku Math. Journ. Vol. 19, No. 1, 1967.