Annales Mathematicae et Informaticae
47 (2017) pp. 141–148
http://ami.uni-eszterhazy.hu

A combinatorial generalization of the gcd-sum function using a generalized Möbius function

Vichian Laohakosol[a], Pinthira Tangsupphathawat[b]

[a]Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand
fscivil@ku.ac.th

[b]Department of Mathematics, Faculty of Science and Technology, Phranakorn Rajabhat University, Bangkok 10220, Thailand
t.pinthira@hotmail.com

Submitted September 4, 2017 — Accepted November 30, 2017

Abstract

Using the Souriau-Hsu-Möbius function with a natural parameter, a generalized Cesáro formula which is an extension of the classical gcd-sum formula is derived. The formula connects a combinatorial aspect of the generalized Möbius function with the number of integers whose prime factors have sufficiently high powers.

Keywords: gcd-sum function, Souriau-Hsu-Möbius function

MSC: 11A25

1. Introduction

Let $\mathcal{A} := \{F : \mathbb{N} \to \mathbb{C}\}$ be the set of complex-valued arithmetic functions. For $F, G \in \mathcal{A}$, their addition and Dirichlet product (or convolution) are defined, respectively, by

$$(F+G)(n) = F(n) + G(n), \quad (F * G)(n) = \sum_{d|\, n} F(d)G(n/d).$$

It is well known, [7, Chapter 7] that $(\mathcal{A}, +, *)$ is commutative ring with identity I, where $I(n) = 1$ if $n = 1$ and $I(n) = 0$ when $n > 1$. The Souriau-Hsu-Möbius function ([9], [2]) is defined, for $\alpha \in \mathbb{C}$, by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where $n = \prod p^{\nu_p(n)}$ denotes the unique prime factorization of n. For some particular values of α, the corresponding Souriau-Hsu-Möbius functions represent certain well-known arithmetic functions, namely,

(i) when $\alpha = 0$, this corresponds to the convolution identity $\mu_0 = I$;

(ii) when $\alpha = 1$, this is the classical Möbius function $\mu_1 = \mu$;

(iii) when $\alpha = -1$, this is the inverse of the Möbius function $\mu_{-1} = \mu^{-1} =: u$, where $u(n) = 1$ ($n \in \mathbb{N}$) is the constant 1 function;

(iv) when $\alpha = -2$, this is the number of divisors function, $\mu_{-2} = d$, [2, p. 75].

Following [11], see also [6], for $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}$, the (k, α)-Euler's totient is defined as an arithmetic function of the form

$$\varphi_{k,\alpha} := \zeta_k * \mu_\alpha \qquad \zeta_k(n) := n^k.$$

When $k = \alpha = 1$, this function is the classical Euler's totient

$$\varphi_{1,1}(n) := \varphi(n) = \zeta_1 * \mu_1(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

which counts the number of integers in $\{1, 2, \ldots, n\}$ that are relatively prime to n. Fixing $k = 1, \alpha = r \in \mathbb{N}$, the corresponding Euler's totient, referred to as *r-Euler totient*, is

$$\varphi_r(n) := \varphi_{1,r}(n) := \zeta_1 * \mu_r(n) = \sum_{d|n} d\mu_r\left(\frac{n}{d}\right).$$

As mentioned in [5, Example 6], the r-Euler totient has the following combinatorial meaning: an integer a is said to be *r^{th}-degree prime* to n (≥ 2), briefly written as $(a, n)_r = 1$, if for each prime divisor p of n, there are integers $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ such that

$$a \equiv a_0 + a_1 p + \cdots + a_{r-1} p^{r-1} \pmod{p^r}.$$

As a convention, we define

$$(a, 1)_r := 1 \quad \text{for any } a \in \mathbb{N}.$$

When $r = 1$, the concept of being r^{th}-degree prime is merely that of being relatively prime.

In order to connect the concept of being r^{th}-degree prime with the r-Euler totient, we introduce another notion. A positive integer n is said to be *r-powerful* if each of its prime factor appears with multiplicity at least r, i.e., $\nu_p(n) \geq r$ for each prime divisor p of n; as a convention, the integer 1 is adopted to be r-powerful for any $r \in \mathbb{N}$. Note that if n is r-powerful, then it is also s-powerful for all $s \in \mathbb{N}$ with $s \leq r$. The following lemma shows that when n is r-powerful, the function $\varphi_r(n)$ counts the number of a's in the set $\{1, 2, \ldots, n\}$ such that $(a, n)_r = 1$; the proof given here is extracted from [11].

Lemma 1.1. *Let $n, r \in \mathbb{N}$, and let $N_r(n)$ denote the number of integers $a \in \{1, 2, \ldots, n\}$ such that $(a, n)_r = 1$. We have :*

1) *The function $F(n) = N_r(n)$ if n is r-powerful and zero otherwise, is multiplicative.*
2) *If n is r-powerful, then $N_r(n) = \varphi_r(n) = n \prod_{p|n} (1 - 1/p)^r$, $N_r(1) = \varphi_r(1) := 1$.*

Proof. 1) Let n be an r-powerful positive integer whose prime factorization is $n = p_1^{e_1} \cdots p_s^{e_s}$. By the Chinese remainder theorem, for any integers $\alpha_1, \ldots, \alpha_s$, there is a unique $a \pmod n$ such that

$$a \equiv \alpha_1 \pmod{p_1^{e_1}}, \ \ldots\ , a \equiv \alpha_s \pmod{p_s^{e_s}}.$$

Conversely, for any $a \pmod n$, there uniquely exist $\alpha_i \pmod{p_i^{e_i}}$ $(i = 1, \ldots, s)$ satisfying the above system of congruences. Thus,

$$(a, n)_r = 1 \iff (a, p_i^{e_i})_r = 1 \text{ holds for every } i \in \{1, \ldots, s\},$$

which shows at once that $N_r(n)$ is a multiplicative function of n.

2) Using part 1), it suffices to check that N_r and φ_r are equal on any prime power p^e with $e \geq r$. Recall that $N_r(p^e)$ is the number of $a \in \{1, 2, \ldots, p^e\}$ such that $(a, p^e)_r = 1$, i.e., such that there are integers $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ satisfying

$$a \equiv a_0 + a_1 p + \cdots + a_{r-1} p^{r-1} \pmod{p^r}.$$

Thus, the number of such $a \pmod{p^r}$ is $(p-1)^r$, and so the total number of such $a \pmod{p^e}$ is $N_r(p^e) = p^{e-r}(p-1)^r$. On the other hand using $e \geq r$, we have

$$\begin{aligned}
\varphi_r(p^e) &= \sum_{d|p^e} d\mu_r(p^e/d) = p^0 \binom{r}{e}(-1)^e + p\binom{r}{e-1}(-1)^{e-1} + \cdots + p^e\binom{r}{0}(-1)^0 \\
&= p^{e-r}\binom{r}{r}(-1)^r + p^{e-r+1}\binom{r}{r-1}(-1)^{r-1} + \cdots + p^e\binom{r}{0}(-1)^0 \\
&= p^{e-r}(p-1)^r = N_r(p^e).
\end{aligned}$$

□

The classical *gcd-sum function* is an arithmetical function defined by

$$g(n) := \sum_{j=1}^{n} \gcd(j, n), \tag{1.1}$$

and the classical *gcd-sum formula* states that

$$g(n) = \sum_{j=1}^{n} \gcd(j, n) = \sum_{d|n} d\varphi\left(\frac{n}{d}\right). \tag{1.2}$$

There have recently appeared quite a number of works related to the gcd-sum function and the gcd-sum formula. In [3], the gcd-sum function (1.1) is shown to be multiplicative, has a polynomial growth, and arises in the context of a lattice point counting problem, while the paper [1] studies the function $\sum_{j=1,\ \gcd(j,n)|d}^{n} \gcd(j,n)$, which is a generalization of the gcd-sum function (1.1).

In [8], the function $\sum_{j=1}^{n} \gcd(j,n)^{-1}$, which counts the orders of a generator of a cyclic group, is studied. In [4], an extended Cesáro formula

$$\sum_{j=1}^{n} f(\gcd(j, n)) = \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) \qquad (f \in \mathcal{A}), \tag{1.3}$$

which is another extension of (1.2), is investigated. Various properties of the gcd-sum function (1.1) and its analogues are surveyed in [10]. Our objective here is to establish yet another generalization of the gcd-sum formula (1.2) by relating the r-Euler totient with the counting of r-powerful integers that are r^{th}-degree prime.

2. Generalized gcd-sum formula

Our generalized gcd-sum formula arises from replacing the usual Euler's totient on the right-hand side of (1.3) by the r-Euler totient, and using its combinatorial meaning to derive its corresponding generalized form. To do so, we need to extend the notion of r^{th}-degree primeness to that of r-gcd.

Definition. Let $r \in \mathbb{N}$, and let $n \in \mathbb{N}$ be r-powerful. For $j \in \mathbb{N}$, the integer g is the *r-gcd* of j and n, denoted by $g := (j, n)_r$, if $g = \gcd(j, n)$ satisfies two additional requirements

1. $\left(\frac{j}{g}, \frac{n}{g}\right)_r = 1$, and
2. n/g is r-powerful.

When $r = 1$, the above definition of r-gcd is identical with the usual greatest common divisor. Let us look at some examples.

Example 1. Let $n = 2^3 \cdot 3^2 = 72$. The divisors of n are $1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72$. Consider $j \in \{1, 2, \ldots, 72\}$. For $r = 1$, we have
$(j, 72)_1 = 1$ when $j \in \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71\}$
$(j, 72)_1 = 2$ when $j \in \{2, 10, 14, 22, 26, 34, 38, 46, 50, 58, 62, 70\}$
$(j, 72)_1 = 3$ when $j \in \{3, 15, 21, 33, 39, 51, 57, 69\}$

$(j, 72)_1 = 4$ when $j \in \{4, 20, 28, 44, 52, 68\}$
$(j, 72)_1 = 6$ when $j \in \{6, 30, 42, 66\}$
$(j, 72)_1 = 8$ when $j \in \{8, 16, 32, 40, 56, 64\}$
$(j, 72)_1 = 9$ when $j \in \{9, 27, 45, 63\}$
$(j, 72)_1 = 12$ when $j \in \{12, 60\}$
$(j, 72)_1 = 18$ when $j \in \{18, 54\}$
$(j, 72)_1 = 24$ when $j \in \{24, 48\}$
$(j, 72)_1 = 36$ when $j \in \{36\}$
$(j, 72)_1 = 72$ when $j \in \{72\}$.

For $r = 2$, we have
$(j, 72)_2 = 1$ when $j \in \{7, 23, 31, 35, 43, 59, 67, 71\}$
$(j, 72)_2 = 2$ when $j \in \{14, 46, 62, 70\}$
$(j, 72)_2 = 8$ when $j \in \{32, 40, 56, 64\}$
$(j, 72)_2 = 9$ when $j \in \{27, 63\}$
$(j, 72)_2 = 18$ when $j \in \{54\}$
$(j, 72)_2 = 72$ when $j \in \{72\}$,
and for the other values of j, the 2-gcd $(j, 72)_2$ are not defined.

For $r = 3$, we have $(j, 72)_3 = 9$ when $j \in \{63\}$, $(j, 72)_3 = 72$ when $j \in \{72\}$, and for the other values of j, the 3-gcd $(j, 72)_3$ are not defined.

For $r \geq 4$, we have $(j, 72)_3 = 72$ when $j \in \{72\}$, and the r-gcd $(j, 72)_r$ are not defined for the remaining j's.

Our next lemma connects the r-gcd with the r-Euler totient.

Lemma 2.1. *Let $r, n \in \mathbb{N}$ with n being r-powerful, and let*

$$A_{r,d}(n) := \{a \in \{1, 2, \ldots, n\}\ ;\ (a, n)_r = d\}.$$

Then

$$|A_{r,d}(n)| = \varphi_r\left(n/d\right).$$

Proof. The result follows at once from the observation that

$$a \in \{1, 2, \ldots, n\} \quad \text{and} \quad (a, n)_r = d,$$

if and only if $\frac{a}{d} \in \left\{1, 2, \ldots, \frac{n}{d}\right\}$, $\left(\frac{a}{d}, \frac{n}{d}\right)_r = 1$, $\frac{n}{d}$ is r-powerful, and the set of elements on the right-hand side has cardinality $\varphi_r\left(n/d\right)$. □

We now state and prove our generalized Cesáro formula.

Theorem 2.2. *Let $r, n \in \mathbb{N}$ with n being r-powerful. For an arithmetic function f, we have the following generalized Cesáro formula*

$$\sum_{j=1}^{n} f((j, n)_r) = \sum_{d\,|_r\,n} f(d)\varphi_r\left(\frac{n}{d}\right). \tag{2.1}$$

where the symbol $d\,|_r\,n$ in the summation on the right-hand side indicates that the sum extends over all the divisors d for which n/d is r-powerful.

In particular, taking $f(n) = n$, the generalized Cesáro formula becomes the generalized gcd-sum formula

$$\sum_{j=1}^{n}(j,n)_r = \sum_{d\,|_r\,n} d\varphi_r\left(\frac{n}{d}\right). \tag{2.2}$$

Proof. Writing $(j,n)_r = d$, using the above notation and Lemma 2.1, we get

$$\sum_{j=1}^{n} f\left((j,n)_r\right) = \sum_{j=1}^{n}\sum_{(j,n)_r=d} f(d) = \sum_{d\,|_r\,n} f(d)\left|A_{r,d}(n)\right| = \sum_{d\,|_r\,n} f(d)\varphi_r\left(\frac{n}{d}\right). \quad \square$$

When $r = 1$, the generalized Cesáro formula is simply the classical Cesáro formula, and its representation via generalized Möbius function becomes a Dirichlet product of two arithmetic functions, viz.,

$$\sum_{j=1}^{n} f((j,n)_1) = \sum_{d\,|_1\,n} f(d)\varphi_1\left(\frac{n}{d}\right) = (f*\varphi)(n).$$

Example 2. Continuing from Example 1, let $n = 2^3\cdot 3^2 = 72$.

For $r = 1$, we have

$$\{d\in\{1,2,\ldots,72\}\,;\, d\,|_1\,72\} = \{1,2,3,4,6,8,9,12,18,24,36,72\},$$

and the values of $\varphi_1\,(72/d)$ with $d\,|_1\,72$ are

$$\begin{aligned}&\varphi_1(72)=24, \varphi_1(36)=12, \varphi_1(24)=8, \varphi_1(18)=6, \varphi_1(12)=4, \varphi_1(9)=6,\\ &\varphi_1(8)=4, \varphi_1(6)=2, \varphi_1(4)=2, \varphi_1(3)=2, \varphi_1(2)=1, \varphi_1(1)=1.\end{aligned}$$

Using Example 1, the left-hand side of (2.1) is

$$\begin{aligned}\sum_{j=1}^{n} f((j,n)_1) &= f(1)\times 24 + f(2)\times 12 + f(3)\times 8 + f(4)\times 6 + f(6)\times 4\\ &\quad + f(8)\times 6 + f(9)\times 4 + f(12)\times 2 + f(18)\times 2 + f(24)\times 2\\ &\quad + f(36)\times 1 + f(72)\times 1\\ &= \sum_{d\,|_1\,n} f(d)\varphi_1\left(\frac{n}{d}\right)\end{aligned} \tag{2.3}$$

which agrees with the theorem.

For $r = 2$, we have $\{d\in\{1,2.\ldots,72\}\,;\, d\,|_2\,72\} = \{1,2,8,9,18,72\}$, and the values of $\varphi_2\,(72/d)$ with $d\,|_2\,72$ are

$$\varphi_2(72)=8, \varphi_2(36)=4, \varphi_2(9)=4, \varphi_2(8)=2, \varphi_2(4)=1, \varphi_2(1)=1.$$

Using Example 1, the left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_2) = f(1)\times 8 + f(2)\times 4 + f(8)\times 4 + f(9)\times 2 + f(18)\times 1 + f(72)\times 1$$
$$= \sum_{d\,|_2\, n} f(d)\varphi_2\left(\frac{n}{d}\right). \tag{2.4}$$

For $r = 3$, we have $\{d \in \{1, 2. \dots, 72\}\,;\, d\,|_3\, 72\} = \{9, 72\}$, and the values of $\varphi_3\,(72/d)$ with $d\,|_3\, 72$ are $\varphi_3(8) = 1, \varphi_3(1) = 1$. Using Example 1, the left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_3) = f(9)\times 1 + f(72)\times 1 = \sum_{d\,|_3\, n} f(d)\varphi_3\left(\frac{n}{d}\right). \tag{2.5}$$

For $r \geq 4$, we have $\{d \in \{1, 2. \dots, 72\}\,;\, d\,|_r\, 72\} = \{72\}$, and the values of $\varphi_3\,(72/d)$ with $d\,|_r\, 72$ is $\varphi_r(1) = 1$. The left-hand side of (2.1) is

$$\sum_{j=1}^{n} f((j,n)_r) = f(72)\times 1 = \sum_{d\,|_r\, n} f(d)\varphi_r\left(\frac{n}{d}\right). \tag{2.6}$$

The formulae such as (2.1)–(2.6) deal with a single r. We end this paper by remarking that such formulae can be absorbed into one single formula. For a positive integer n whose prime representation is $n = p_1^{\nu_1(n)} p_2^{\nu_2(n)} \cdots p_t^{\nu_t(n)}$, let

$$\nu(n) := \max\{\nu_i(n); 1 \leq i \leq t\}.$$

Corollary 2.3. *For an arithmetic function f, we have the following generalized Cesáro formula*

$$\sum_{r=1}^{\nu(n)} \sum_{j=1}^{n} f((j,n)_r) = \sum_{r=1}^{\nu(n)} \sum_{d\,|_r\, n} f(d)\varphi_r\left(\frac{n}{d}\right).$$

Acknowledgements. The first author is supported by Center for Advanced Studies in Industrial Technology and Faculty of Science, Kasetsart University. The second author is supported by Phranakhon Rajabhat University.

References

[1] Abel, U., Awan W., Kushnirevych, V., A generalization of the gcd-sum function, *J. Integer Sequences* Vol. 16 (2013), Article 13.6.7.

[2] Brown, T. C., Hsu, L. C., Wang, J., Shiue, P. J. S., On a certain kind of generalized number-theoretical Möbius function, *Math. Sci.* Vol. 25 (2000), 72–77.

[3] Broughan, K. A., The gcd-sum function, *J. Integer Sequences* Vol. 4 (2001), Article 01.2.2.

[4] Haukkanen, P., On a gcd-sum function, *Aequationes Math.* Vol. 76 (2008), 168–178.

[5] Hsu, L. C., Wang, J., Some Möbius-type functions and inversions constructed via difference operators, *Tamkang J. Math.* Vol. 29 (1998), 89–99.

[6] Pabhapote, N., Laohakosol V., Combinatorial aspects of the generalized Euler's totient, *Intern. J. Math. Math. Sci.* (2010), Article ID 648165, 15 pages.

[7] Redmond, D., Number Theory, An Introduction, Marcel Dekker, New York, 1996.

[8] Sándor, J., Kramer, A. V., Über eine zahlentheoretische Funktion, *Math. Moravica* Vol. 3 (1999), 53–62.

[9] Souriau, Jean-Marie, Généralisation de certaines formules arithmétiques d' inversion. Applications, *Revue Scientific (Rev. Rose Illus.)* Vol. 82 (1944), 204–211.

[10] Tóth, L., A survey of gcd-sum functions, *J. Integer Sequences* Vol. 13 (2010), Article 10.8.1.

[11] Wang, J., Hsu, L. C., On certain generalized Euler-type totients and Möbius-type functions, Dalian University of Technology, China, preprint.