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Determining special roots of quaternion polynomials

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Abstract

In this paper we determine the sets of spherical roots, real roots, isolated complex roots, pure imaginary quaternion roots and roots in $\mathbb{R} + \mathbb{R}\mathbf{j}$ and $\mathbb{R} + \mathbb{R}\mathbf{k}$ of a quaternion polynomial $Q(t)$ by corresponding these sets to the sets of real or complex roots of some real or complex polynomials determined by $Q(t)$. Thus, the counting and classifying methods for such polynomials can be used for the counting and classifying of the aforementioned roots of quaternion polynomials.

Keywords: Quaternion polynomial; Spherical Root; Isolated Root; Complex Root.

MSC: 12E15; 11R52; 16H05.

1. Introduction

The problem of counting and classifying the real/imaginary roots of a given real polynomial has been extensively studied. The classical Sturm's algorithm is an efficient method for determining the numbers of real roots of constant coefficient polynomials, but very inconvenient for those with symbolic coefficients. On the other hand, since the complete root classification of a parametric polynomial has been applied in studies of ordinary differential equations, of integral equations, of mechanics problems, and to real quantifier elimination (see [16]), several methods have been developed to fulfil this task [9, 17, 26, 27]. As far as we know, similar results do not exist for the roots of polynomials

$$Q(t) = a_0t^n + a_1t^{n-l} + \cdots + a_n$$

with coefficients $a_0, \ldots, a_n$ lying in the skew field of quaternions $\mathbb{H}$.

The fundamental theorem of algebra holds for quaternion polynomials [18, 5, 8, 14]. Algorithms for the computation of roots of a quaternion polynomial and its expression as a product of linear factors have been investigated in several papers [6, 7, 10, 11, 12, 13, 14, 19, 20, 21, 22, 24, 25]. Recently, a method for finding the pure imaginary quaternion roots of a quaternion polynomial was given [1]. Furthermore, necessary and sufficient conditions for a quaternion polynomial to have a special kind of root were obtained [4]. Note that a special class of space curves, the so-called Pythagorean hodograph curves, may be generated by quaternion polynomials [2, 3], and as it is has been studied in [3, Chapter 6], such a curve is generated by another curve of lower degree if and only if its associated quaternion polynomial has a complex root.

In this paper we determine the sets of spherical roots, real roots, isolated complex roots, pure imaginary quaternion roots and roots in $\mathbb{R} + \mathbb{R}\mathbf{j}$ and $\mathbb{R} + \mathbb{R}\mathbf{k}$ of a quaternion polynomial $Q(t)$ by defining a bijection from each of these sets onto the sets of real or complex roots of some real or complex polynomials which are determined by $Q(t)$. So the counting and classifying methods for real and complex polynomials can be used for counting and classifying the aforementioned roots of quaternion polynomials. Our method for the study of pure imaginary quaternion roots has the same initial point as Chapman's approach [1, Chapter 4, Section 1.3], but is quite different in the sequel, since we use simpler equations and we determine exactly the sets of spherical and isolated pure imaginary quaternion roots.

The paper is organized as follows. In Section 2, we recall basic facts about quaternions and quaternion polynomials. Section 3 is devoted to the study of pure imaginary quaternion roots of quaternion polynomials. The spherical roots, real roots, complex isolated roots, and roots in $\mathbb{R}+\mathbb{R}\mathbf{j}$ and $\mathbb{R}+\mathbb{R}\mathbf{k}$ are studied in Section 4. Finally, in Section 5, we illustrate our results with three examples.

2. Quaternions

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively. We denote by $\mathbb{H}$ the skew field of real quaternions. Its elements are of the form $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The *conjugate* of q is defined as $\bar{q} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. The *real* and the *imaginary part* of q are $\operatorname{Re} q = x_0$ and $\operatorname{Im} q = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, respectively. If $\operatorname{Re} q = 0$, then q

is called *pure imaginary quaternion.* The *norm* $|q|$ of q is defined to be the quantity

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Two quaternions q and q' are said to be *congruent* or *equivalent,* written $q \sim q'$, if there is $w \in \mathbb{H} \setminus \{0\}$ such that $q' = wqw^{-1}$. By [28], we have $q \sim q'$ if and only if $\operatorname{Re} q = \operatorname{Re} q'$ and $|q| = |q'|$. The *congruence class* of q is the set

$$[q] = \{q' \in \mathbb{H}/\ q' \sim q\} = \{q' \in \mathbb{H}/\ \operatorname{Re} q = \operatorname{Re} q',\ |q| = |q'|\}.$$

Note that every class $[q]$ contains exactly one complex number z and its conjugate $\bar{z}$, which are $x_0 \pm \mathbf{i}\sqrt{x_1^2 + x_2^2 + x_3^2}$.

Let $\mathbb{H}[t]$ be the polynomial ring in the variable t over $\mathbb{H}$. Every polynomial $f(t) \in \mathbb{H}[t]$ is written as $a_0 t^n + a_1 t^{n-l} + \cdots + a_n$ where n is an integer ≥ 0 and $a_0, \ldots, a_n \in \mathbb{H}$ with $a_0 \neq 0$. The addition and the multiplication of polynomials are defined in the same way as the commutative case, where the variable t is assumed to commute with quaternion coefficients [15, Chapter 5, Section 16]. For every $q \in \mathbb{H}$ we define the evaluation of $f(t)$ at q to be the element

$$f(q) = a_0 q^n + a_1 q^{n-l} + \cdots + a_n.$$

Note that it is not in general a ring homomorphism from $\mathbb{H}[t]$ to $\mathbb{H}$.

We say that a quaternion q is a zero or a root of $f(t)$ if $f(q) = 0$. The polynomial $B(t) \in \mathbb{H}[t]$ is called a *right factor* of $Q(t)$ if there exists $C(t) \in \mathbb{H}[t]$ such that $Q(t) = C(t)B(t)$. Note that q is a root of $f(t)$ if and only if $t - q$ is a right factor of $Q(t)$, i.e. there exists $g(t) \in \mathbb{H}[t]$ such that $f(t) = g(t)(t - q)$ [15, Proposition 16.2].

Let q be a root of $f(t)$. If q is not real and has the property that $f(z) = 0$ for all $z \in [q]$, then we will say that q generates a *spherical root.* For short, we will also say that q is, rather than generates, a spherical root. If q is real or does not generate a spherical zero, it is called an isolated root. If two elements of a class are roots of $f(t)$, then [10, Theorem 4] implies that all elements of this class are zeros of $f(t)$. Therefore, since every congruence class contains exactly one complex number z and its conjugate $\bar{z}$, the pairs of complex numbers $\{z, \bar{z}\}$ which are roots of $f(t)$ determine all spherical roots of $f(t)$.

3. Pure imaginary quaternion roots

In this section we determine the set of pure imaginary quaternion roots of a quaternion polynomial. Let $Q(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$. We set

$$g(t) = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} a_{2m+1}(-1)^m t^m \quad \text{and} \quad h(t) = \sum_{m=0}^{\lfloor n/2 \rfloor} a_{2m}(-1)^m t^m.$$

Let $g_i(t), h_i(t) \in \mathbb{R}[t]$ $(i = 1, 2, 3, 4)$ such that

$$g(t) = g_1(t) + g_2(t)\mathbf{i} + g_3(t)\mathbf{j} + g_4(t)\mathbf{k},$$

$$h(t) = h_1(t) + h_2(t)\mathbf{i} + h_3(t)\mathbf{j} + h_4(t)\mathbf{k}.$$

We denote by $E(t)$ the greatest common divisor of polynomials $g_1(t), \dots, g_4(t)$, $h_1(t), \dots, h_4(t)$. Further, we consider the polynomials

$$F(t) = (g_1(t)^2 + g_2(t)^2 + g_3(t)^2 + g_4(t)^2)t - (h_1(t)^2 + h_2(t)^2 + h_3(t)^2 + h_4(t)^2)$$

and

$$G(t) = g_1(t)h_1(t) + g_2(t)h_2(t) + g_3(t)h_3(t) + g_4(t)h_4(t).$$

Let $L(t) = \gcd(F(t), G(t))$. Note, that $E(t)$ divides $L(t)$.

Theorem 3.1. *Let $\mathcal{E}$ be the set of the positive roots of $E(t)$ and $\mathcal{L}$ the set of the positive roots of $L(t)$ which are not roots of $E(t)$. We denote by S and I the sets of distinct spherical and isolated pure imaginary quaternion roots of $Q(t)$, respectively. Then the maps*

$$\begin{aligned} \sigma\colon \mathcal{E} &\longrightarrow S, \\ N &\longmapsto \{x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}/\ x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = N\} \end{aligned}$$

and

$$\begin{aligned} \tau\colon \mathcal{L} &\longrightarrow I, \\ N &\longmapsto -g(N)^{-1}h(N) \end{aligned}$$

are bijective.

Proof. Let $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, with $x \neq 0$, be a pure imaginary quaternion which is a root of $Q(t)$. Then we have $x^2 = -(x_1^2 + x_2^2 + x_3^2) = -|x|^2$ and setting $N = x_1^2 + x_2^2 + x_3^2$, we get $x^2 = -N$. Thus, the equality $Q(x) = 0$ implies $g(N)x + h(N) = 0$. Further, since $x \neq 0$, we have $g(N) = 0$ if and only if $h(N) = 0$.

Suppose that x defines a spherical root of $Q(t)$. It follows that every $y \in [x]$ is a root of $Q(t)$. Further, y is a pure imaginary quaternion with $|x| = |y|$, and so $y^2 = -|y|^2 = -|x|^2 = x^2$, whence $g(N)y = -h(N)$. Thus, for every $y \in [x]$ we have $g(N)y = -h(N)$, whence we deduce $g(N) = h(N) = 0$. Since N is a real number, we have $E(N) = 0$, and so $N \in \mathcal{E}$. Conversely, if $N \in \mathcal{E}$, then $g(N) = h(N) = 0$. Thus, for every pure imaginary quaternion x with $|x| = \sqrt{N}$, we have $Q(x) = g(N)x + h(N) = 0$. Hence, $[x]$ is a spherical root of $Q(t)$. Therefore, σ is a bijection.

We have that $E(t)$ divides $F(t)$ and $G(t)$, whence we have that $E(t)$ divides $L(t)$. Suppose now that $L(t) \neq E(t)$. Let x be an isolated pure imaginary quaternion root of $Q(t)$ with $g(N)h(N) \neq 0$. Then, we get $g(N)x + h(N) = 0$, whence we get $|g(N)|^2|x|^2 = |h(N)|^2$, and so, N is a root of $F(t)$. Furthermore, we have

$$x = -g(N)^{-1}h(N) = -\overline{g(N)}h(N)/|g(N)|^2$$

and, since $\operatorname{Re} x = 0$, we get $G(N) = 0$. Hence N is a real positive root of $L(t)$, and so $N \in \mathcal{L}$. Conversely, suppose that $N \in \mathcal{L}$. Set $x = -\overline{g(N)}h(N)/|g(N)|^2$.

Since $G(N) = 0$, we have $\operatorname{Re} x = 0$. Furthermore, N is a root of $F(t)$, and so, we get $N = |h(N)|^2/|g(N)|^2 = |x|^2$. Thus, x is a purely imaginary quaternion with $x^2 = -|x|^2 = -N$, and hence $Q(x) = g(N)x + h(N) = 0$. Therefore, τ is a bijection. $\square$

Corollary 3.2. *The numbers of spherical roots of $Q(t)$ is equal to the number of positive roots of $E(t)$, and the number of isolated pure imaginary quaternion roots of $Q(t)$ is equal to the number of positive roots of $L(t)$ which are not roots of $E(t)$.*

Corollary 3.3. *Let $l_1, \dots, l_\nu$ be the positive real roots of $L(t)$ satisfying $g(l_i)h(l_i) \neq 0$ $(i = 1, \dots, \nu)$. Then, the quaternions $q_i = -g(l_i)^{-1}h(l_i)$ $(i = 1, \dots, \nu)$ are all the isolated pure quaternion roots of $Q(t)$.*

Proof. The map τ of Theorem 3.1 is a bijection. Further, the roots of $L(t)$ which are also roots of $E(t)$ are the roots ρ such that $g(\rho) = h(\rho) = 0$. The other roots satisfy $g(\rho) \neq 0$ and $h(\rho) \neq 0$. Thus, these roots ρ yield the isolated pure imaginary quaternion roots of $Q(t)$ which are the quaternions $-g(\rho)^{-1}h(\rho)$. $\square$

4. Spherical roots and roots in $\mathbb{C}$, $\mathbb{R} + \mathbb{R}\mathbf{j}$ and $\mathbb{R} + \mathbb{R}\mathbf{k}$

In this section we study first the sets of spherical roots, complex isolated roots and real roots of a quaternion polynomial. Let $Q(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ be a monic polynomial of degree ≥ 1. Write $Q(t) = f_1(t) + f_2(t)\mathbf{i} + g_1(t)\mathbf{j} + g_2\mathbf{k}$, where $f_1(t), f_2(t), g_1(t), g_2(t) \in \mathbb{R}[t]$, and let $\Delta(t) = \gcd(f_1(t), f_2(t), g_1(t), g_2(t))$. Set $f(t) = f_1(t) + f_2(t)\mathbf{i}$, $g(t) = g_2(t) + g_1(t)\mathbf{i}$, $E(t) = \gcd(f(t), g(t))$ and $\Lambda(t) = E(t)/\Delta(t)$.

Theorem 4.1. *a) The set of real roots of $Q(t)$ coincides with the set of real roots of $\Delta(t)$.*
b) The spherical roots of $Q(t)$ are represented by the pairs of complex conjugate roots of $\Delta(t)$.
c) The set of isolated complex roots of $Q(t)$ is equal to the set of roots of $\Lambda(t)$.

Proof. a) Let $x \in \mathbb{R}$. Then we have $Q(x) = 0$ if and only if

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0$$

which is equivalent to $\Delta(x) = 0$. It follows that the set of real roots of $Q(t)$ is the same with the set of real roots of $\Delta(t)$.

b) We have $Q(t) = f(t) + \mathbf{k}g(t)$. Let $z \in \mathbb{C}$. We have $Q(z) = 0$ if and only if $f(z) + \mathbf{k}g(z) = 0$ which is equivalent to $f(z) = g(z) = 0$. Suppose now that $Q(t)$ has a spherical root q. Let z and $\bar{z}$ be the only complex numbers of the class of q. Then we have $Q(z) = Q(\bar{z}) = 0$, whence we get $f(z) = f(\bar{z}) = 0$ and $g(z) = g(\bar{z}) = 0$. It follows that the real polynomial $(t - z)(t - \bar{z})$ divides $f(z)$ and $g(z)$ and hence $(t - z)(t - \bar{z})$ divides the polynomials $f_1(t), f_2(t), g_1(t), g_2(t)$. Therefore z and $\bar{z}$ is a pair of conjugate complex root of $\Delta(t)$. Conversely, suppose z and $\bar{z}$ is a pair of conjugate complex root of $\Delta(t)$. It follows that z and $\bar{z}$ are roots of $f(t)$ and $g(t)$ and hence of $Q(t)$. Therefore, the class of z is a spherical root of $Q(t)$. So, there is a bijection between the spherical roots of $Q(t)$ and the pairs of complex conjugate roots of $\Delta(t)$.

c) Suppose that $C(t) \in \mathbb{C}[t]$. The polynomial $C(t)$ is a right factor of $Q(t)$ if and only if there is $A(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ such that $Q(t) = A(t)C(t)$. This happens if and only if $C(t)$ divides $f(t)$ and $g(t)$ which is equivalent to the fact that $C(t)$ divides $E(t)$. Thus, $C(t)$ is a right factor of $Q(t)$ if and only if $C(t)$ divides $E(t)$. In case where $C(t) \in \mathbb{R}[t]$, we similarly deduce that $C(t)$ is a right factor of $Q(t)$ if and only if $C(t)$ divides $\Delta(t)$. Thus, we have that $\Delta(t)$ divides $E(t)$ and the polynomial $\Lambda(t) = E(t)/\Delta(t)$ has no real factor. Suppose that z is a complex no real isolated root of $Q(t)$. Then its conjugate $\bar{z}$ is not a root of $Q(t)$ and so, $\bar{z}$ is not a root of $E(t)$. It follows that z is a root of $\Lambda(t)$. Conversely, suppose that z is a root of $\Lambda(t)$. If its conjugate $\bar{z}$ is also a root of $\Lambda(t)$, then $(t-z)(t-\bar{z})$ is a real factor of $\Lambda(t)$ which is a contradiction. Then, $\bar{z}$ is not a root of $E(t)$, and so, it is not a root of $Q(t)$. Hence, z is an isolated complex no real root of $Q(t)$. Thus, the complex no real isolated roots of $Q(t)$ are precisely the roots of $\Lambda(t)$. □

Next, we deal with the roots of $Q(t)$ in $\mathbb{R} + \mathbb{R}\mathbf{j}$ and $\mathbb{R} + \mathbb{R}\mathbf{k}$. Set $\bar{f}(t) = f_1(t) + g_1(t)\mathbf{j}$, $\bar{g}(t) = g_2(t) - f_2(t)\mathbf{j}$ and $\bar{E}(t) = \gcd(\bar{f}(t), \bar{g}(t))$. Further, we put $\tilde{f}(t) = f_1(t) + g_2(t)\mathbf{k}$, $\tilde{g}(t) = g_1(t) - f_2(t)\mathbf{k}$ and $\tilde{E}(t) = \gcd(\tilde{f}(t), \tilde{g}(t))$.

Theorem 4.2. *a) The set of roots of $Q(t)$ in $\mathbb{R} + \mathbb{R}\mathbf{j}$ is equal to the set of roots of $\bar{E}(t)$.*
b) The set of roots of $Q(t)$ in $\mathbb{R} + \mathbb{R}\mathbf{k}$ is equal to the set of roots of $\tilde{E}(t)$.

Proof. For (a), we write $Q(t) = \bar{f}(t) + \mathbf{k}\bar{g}(t)$. Let $x \in \mathbb{R} + \mathbb{R}\mathbf{j}$. Then $Q(x) = 0$ if and only if $\bar{f}(x) = \bar{g}(x) = 0$ which is equivalent to $\bar{E}(x) = 0$. For (b), we write $Q(t) = \tilde{f}(t) + \mathbf{i}\tilde{g}(t)$, and similarly we deduce the result. □

5. Examples

In this section we give three examples using the results of previous sections.

Example 5.1. By [1, Chapter 4, Example 1.4.1], the roots of the polynomial

$$P(t) = t^3 + (2 + \mathbf{k})t + \mathbf{i} - \mathbf{j}.$$

are the pure imaginary quaternions $\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.

We shall compute the pure quaternion roots of $P(t)$ using Corollary 3.3. We follow the notations of Section 3. We have $g(t) = -t + 2 + \mathbf{k}$ and $h(t) = \mathbf{i} - \mathbf{j}$. Thus, we get $g_1(t) = -t + 2$, $g_2(t) = g_3(t) = 0$, $g_4(t) = 1$ and $h_1(t) = h_4(t) = 0$, $h_2(t) = 1$, $h_3(t) = -1$. Hence, the greatest common divisor $E(t)$ of these polynomial is 1. It follows that $P(t)$ has not a spherical pure imaginary quaternion root. Next, we obtain the polynomials

$$F(t) = ((-t + 2)^2 + 1)t - (1 + 1) = t^3 - 4t^2 + 5t - 2 \quad \text{and} \quad G(t) = 0.$$

Then $L(t) = \gcd(F(t), G(t)) = t^3 - 4t^2 + 5t - 2$. The roots of $L(t)$ are 1 and 2. Next, we compute:

$$-g(1)^{-1}h(1) = -(1+\mathbf{k})^{-1}(\mathbf{i}-\mathbf{j}) = \mathbf{j}, \quad -g(2)^{-1}h(2) = -\mathbf{k}^{-1}(\mathbf{i}-\mathbf{j}) = \mathbf{j}+\mathbf{i}.$$

Hence, the isolated pure imaginary quaternion roots of $P(t)$ are $\mathbf{j}$ and $\mathbf{i}+\mathbf{j}$.

Example 5.2. According to [12], the polynomial

$$Q(t) = t^6 + \mathbf{j}\,t^5 + \mathbf{i}\,t^4 - t^2 - \mathbf{j}\,t - \mathbf{i}$$

has the four isolated roots

$$t_1 = 1, \quad t_2 = -1, \quad t_3 = \frac{1}{2}(1-\mathbf{i}-\mathbf{j}-\mathbf{k}), \quad t_4 = \frac{1}{2}(-1+\mathbf{i}-\mathbf{j}-\mathbf{k})$$

and the spherical root generated by $t_5 = \mathbf{i}$.

Following the notations of Section 4, we have:

$$f_1(t) = t^6 - t^2, \quad f_2(t) = t^4 - 1, \quad g_1(t) = t^5 - t, \quad g_2(t) = 0.$$

Then

$$\Delta(t) = \gcd(f_1(t), f_2(t), g_1(t), g_2(t)) = t^4 - 1.$$

By Theorem 4.1, we have that $Q(t)$ has the real roots ± 1 and one spherical root defined by $\mathbf{i}$.

Example 5.3. We shall compute the roots of the polynomial

$$R(t) = t^4 - (2+\mathbf{k})t^3 + (3+\mathbf{j}+2\mathbf{k})t^2 - 2(1+\mathbf{j}+\mathbf{k})t + 2(1+\mathbf{j}).$$

Following the notations of Section 4, we write $R(t) = f(t) + \mathbf{k}g(t)$, where

$$f(t) = t^4 - 2t^3 + 3t^2 - 2t + 2, \quad g(t) = -t^3 + (2+\mathbf{i})t^2 - 2(1+\mathbf{i})t + 2\mathbf{i}.$$

We have:

$$E(t) = \gcd(f(t), g(t)) = t^3 - (2+\mathbf{i})t^2 + 2(1+\mathbf{i})t - 2\mathbf{i}.$$

Next, we write $R(t) = f_1(t) + f_2(t)\mathbf{i} + g_1(t)\mathbf{j} + g_2(t)\mathbf{k}$, where

$$f_1(t) = t^4 - 2t^3 + 3t^2 - 2t + 2,\ f_2(t) = 0,\ g_1(t) = t^2 - 2t + 2,\ g_2(t) = -t^3 + 2t^2 - 2t.$$

Then, we have:

$$\Delta(t) = \gcd(f_1(t), f_2(t), g_1(t), g_2(t)) = t^2 - 2t + 2.$$

The roots of $\Delta(t)$ are the complex numbers $1 \pm \mathbf{i}$ which define a spherical root of $R(t)$. Further, we get:

$$\Lambda(t) = \frac{E(t)}{\Delta(t)} = t - \mathbf{i},$$

and so $\mathbf{i}$ is a complex isolated root of $R(t)$.

Next, we shall compute the pure quaternion roots of $R(t)$. Following the notations of Section 3, we compute:

$$g(t) = -2 + 2t - 2\mathbf{j} + (-2 + t)\mathbf{k}, \quad h(t) = 2 - 3t + t^2 + (2 - t)\mathbf{j} - 2t\mathbf{k}.$$

It follows:

$$F(t) = 28t - 30t^2 + 11t^3 - 8 - t^4 \quad \text{and} \quad G(t) = -4t + 6t^2 - 2t^3.$$

We have:

$$L(t) = \gcd(F(t), G(t)) = t^2 - 3t + 2 = (t - 1)(t - 2).$$

We obtain $-g(1)^{-1}h(1) = \mathbf{i}$ and $-g(2)^{-1}h(2) = \mathbf{k} + \mathbf{i}$. Accordingly to [6], the polynomial $R(t)$ has not other roots. Thus, $R(t)$ has the isolated roots $\mathbf{i}$ and $\mathbf{i} + \mathbf{k}$ and the spherical root defined by $1 + \mathbf{i}$.

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