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Twelve subsets of permutations enumerated as maximally clustered permutations

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Abstract

The problem of avoiding a single pattern or a pair of patterns of length four by permutations has been well studied. Less is known about the avoidance of three 4-letter patterns. In this paper, we show that the number of members of S_n avoiding any one of twelve triples of 4-letter patterns is given by sequence A129775 in OEIS, which is known to count maximally clustered permutations. Numerical evidence confirms that there are no other (non-trivial) triples of 4-letter patterns giving rise to this sequence and hence one obtains the largest $(4,4,4)$-Wilf-equivalence class for permutations. We make use of a variety of methods in proving our result, including recurrences, the kernel method, direct counting, and bijections.

Keywords: pattern avoidance; Wilf-equivalence; kernel method; maximally clustered permutations

MSC: 05A15, 05A05

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1. Introduction

1.1. Background

The pattern avoidance question is an extensively studied problem in enumerative and algebraic combinatorics. It has its origins with Knuth [5] and Simion and Schmidt [8] who considered the problem on permutations and enumerated the number of members of S_n avoiding a particular element or subset, respectively, of S_3. Since then the problem has been addressed on several other discrete structures, such as compositions, k-ary words, and set partitions; see, e.g., the texts [3, 7] and references contained therein. Here, we provide further enumerative results concerning the classical avoidance problem on permutations.

Members of S_n avoiding a single 4-letter pattern have been well studied (see, e.g., [9, 10, 11]). There are 56 symmetry classes of pairs of 4-letter patterns, for all but 8 of which the avoiders have been enumerated [2]. Less is known about the 317 symmetry classes of triples of 4-letter patterns. In this paper, we show that precisely 12 of them have the counting sequence of maximally clustered permutations (sequence A129775 in OEIS), which has generating function

$$\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}} = 1+\frac{x}{2-x-C(x)},$$

where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. Based on numerical evidence, this corresponds to the largest $(4,4,4)$-Wilf-equivalence class for permutations.

A computer check of initial terms eliminates all but 12 candidate classes for this counting sequence. We next recall basic terminology, review some standard results, list a representative triple π_i, $i=1,2,\ldots,12$, for each class, and state the main result. Then, in Section 2, we treat each π_i in turn. Our methods include recurrences, the kernel method for solving them, direct counting, and bijections.

1.2. Notation, terminology and main result

Let $\pi=\pi_1\pi_2\cdots\pi_n\in S_n$ and $\tau\in S_k$ be two permutations. We say that π *contains* τ if there exists a subsequence $1\le i_1<i_2<\cdots<i_k\le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is *order-isomorphic* to τ; in this context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ*-avoiding*, if such a subsequence fails to exist. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π *avoids* T if π avoids every $\tau\in T$; the corresponding subset of S_n is denoted $S_n(T)$. Two sets of patterns T and T' are said to be *Wilf-equivalent* if $|S_n(T)|=|S_n(T')|$ for all $n\ge 0$.

The maximally clustered permutations are those that avoid 3421, 4312 and 4321, and this triple is in the same symmetry class as π_3 in Theorem 1.1 below. (See [1], where a different proof is given in this particular case.) Here, symmetry refers to the action of the dihedral group of order 8 generated by the operations reverse, complement, and inverse on permutation patterns. Two pattern sets so related obviously have equinumerous avoiders, in short, are trivially Wilf-equivalent.

For a permutation p on a set of positive integers, the standardization of p, denoted $\mathrm{St}(p)$, is obtained by replacing the smallest entry of p by 1, the next smallest by 2, and so on. Thus π avoids τ iff no subsequence of π has standardization equal to τ. It is well known [8] that, for each 3-letter pattern τ, $|S_n(\tau)|$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Throughout, we use $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ to denote the generating function $\sum_{n\geq 0} C_n x^n$.

Theorem 1.1 (Main Theorem)**.** *Define*

$$\begin{aligned}
\pi_1 &= \{1324, 2134, 2143\}, & \pi_2 &= \{1243, 1324, 2134\}, & \pi_3 &= \{1234, 1243, 2134\},\\
\pi_4 &= \{2314, 2341, 2413\}, & \pi_5 &= \{2314, 2413, 2431\}, & \pi_6 &= \{1423, 3142, 4132\},\\
\pi_7 &= \{1324, 1342, 3142\}, & \pi_8 &= \{1324, 1342, 3124\}, & \pi_9 &= \{1324, 1342, 2314\},\\
\pi_{10} &= \{1324, 1432, 2431\}, & \pi_{11} &= \{1423, 1432, 4132\}, & \pi_{12} &= \{1342, 1423, 4123\}.
\end{aligned}$$

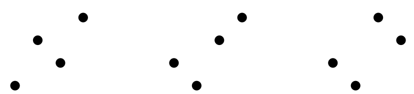
Then, for all $j = 1, 2, \ldots, 12$,

$$\sum_{n\geq 0} \#S_n(\pi_j)x^n = \frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}.$$

2. Proof of main theorem

2.1. Class 1

$\pi_1 = \{1324, 2134, 2143\}$, with graphical representation



Let $A_n = S_n(\pi_1)$. Define $a_n = \#A_n$ and $a_n(i_1, \ldots, i_s)$ to be the number of permutations $\sigma_1\sigma_2\cdots\sigma_n \in A_n$ such that $\sigma_1\sigma_2\cdots\sigma_s = i_1 i_2 \cdots i_s$. Then we have the following recurrence.

Lemma 2.1. *For all* $1 \leq i \leq n-2$,

$$a_n(i) = 2a_{n-1}(i) + a_{n-2}(i)\delta_{i\leq n-3} + \sum_{j=i+2}^{n-2} C_{n-j}a_{j-1}(i),$$

with $a_n(n-1) = a_n(n) = a_{n-1}$.

Proof. By the definitions, $a_n(n) = a_n(n-1) = a_{n-1}$. If $1 \leq i \leq n-2$, then

$$a_n(i) = a_n(i, i+1) + a_n(i, n) + a_n(i, n-1)\delta_{i\leq n-3} + \sum_{j=i+2}^{n-2} a_n(i, j)$$

$$= 2a_{n-1}(i) + a_n(i, n-1, n)\delta_{i \le n-3} + \sum_{j=i+2}^{n-2} a_n(i, j)$$

$$= 2a_{n-1}(i) + a_{n-2}(i)\delta_{i \le n-3} + \sum_{j=i+2}^{n-2} a_n(i, j).$$

Note that any permutation $\pi = ij\pi' \in A_n$ with $i+2 \le j \le n-2$ can be decomposed as $\pi = ij\alpha\beta$, where each letter of α is greater than each letter of β and α avoids 213 and $i\beta \in A_{j-1}$. Thus, by the fact that the number of permutations of length d that avoid 213 is given by the d-th Catalan number (see [5]), we obtain that $a_n(i, j) = C_{n-j}a_{j-1}(i)$, which completes the proof. □

Define $A_n(v)$ to be the polynomial $\sum_{i=1}^{n} a_n(i)v^{i-1}$. Then Lemma 2.1 can be translated in terms of $A_n(v)$ as

$$\begin{aligned} &A_n(v) - A_{n-1}(1)(v^{n-2} + v^{n-1}) \\ &\quad = 2A_{n-1}(v) + A_{n-2}(v) - 2A_{n-2}(1)v^{n-2} - A_{n-3}(1)v^{n-3} \\ &\quad + \sum_{j=3}^{n-2} C_{n-j}(A_{j-1}(v) - A_{j-2}(1)v^{j-2}). \end{aligned}$$

Note that $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1 + v$. Define $A(x, v) = \sum_{n \ge 0} A_n(v)x^n$. Multiplying the last recurrence by x^n, and summing over $n \ge 3$, yields

$$\begin{aligned} &A(x, v) - \frac{x}{v}(A(xv, 1) - 1) - xA(xv, 1) - 1 \\ &\quad = x(2 + x)(A(x, v) - 1) - x^2(2 + x)A(xv, 1) \\ &\quad + x(C(x) - 1 - x)(A(x, v) - 1 - x) - x^2(C(x) - 1 - x)(A(xv, 1) - 1), \end{aligned}$$

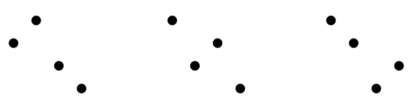
which, upon setting $v = 1$, gives the following result.

Theorem 2.2. *The generating function for the number of permutations of length n that avoid π_1 is given by*

$$\frac{2(1 - 4x)}{2 - 9x + 4x^2 - x\sqrt{1 - 4x}}.$$

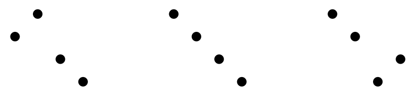
2.2. Class 2

We use the representative triple $\pi_2 := \{X, Y, Z\}$, as illustrated,



$X = 3421$ $Y = 4231$ $Z = 4312$,

compared with



3421 4321 4312 ,

the pattern set π_3 considered in Class 3 below. Note that they differ only in the middle of the middle pattern. Clearly, a permutations avoids π_2 if and only if each of its components does so and the same is true of π_3. So the following result shows that $|S_n(\pi_2)| = |S_n(\pi_3)|$.

Theorem 2.3. *The map "locate the maximal runs of consecutive fixed points and reverse each run" is a bijection from the indecomposable permutations in $S_n(\pi_3)$ to the indecomposable permutations in $S_n(\pi_2)$.*

Proof. As an example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 11 & 1 & 4 & 5 & 6 & 2 & 8 & 9 & 7 & 10 \end{pmatrix}$$
$$\mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 11 & 1 & 6 & 5 & 4 & 2 & 9 & 8 & 7 & 10 \end{pmatrix}.$$

From the characterization of indecomposable π_3-avoiders in Class 3 below, it is clear that the map is one-to-one and into; the only issue is whether it is onto. To show that it is, we investigate the structure of π_2-avoiders.

Lemma 2.4. *Suppose $c > b_1 > b_2 > \cdots > b_r > a$, $r \geq 1$, is a maximal decreasing subsequence of length ≥ 3 in a π_2-avoider p. Then, in the matrix diagram of p, the entries $b_1, b_2, \ldots, b_r$ form the reverse (NW to SE) diagonal of a square bisected by the main diagonal and c is the only entry lying NW of this square and a is the only entry lying SE of it.*

Proof. Consider the rectangles in the matrix determined by the subsequence as shown in Figure 1 for $r \geq 2$ (collapsing some regions covers the case $r = 1$). The gray rectangles are all empty for the indicated reason where M refers to the maximal condition in the hypothesis, and X, Z refer to offending patterns. The entries in the rectangle B are decreasing (else a Y offender is present). Furthermore, since the rest of the row and column containing B is empty, the entries in $b_1 B\, b_r$ must be consecutive and B must be a square of side length $r-2$. Also, the entries in rectangle A consist of $[b_r - 1] \setminus \{a\}$. This means that A is a square of side length $b_r - 1$, and so B is bisected by the main diagonal. Thus, all parts of the lemma have been established. □

It follows from Lemma 2.4 that the mapping is onto and, hence, a bijection. □

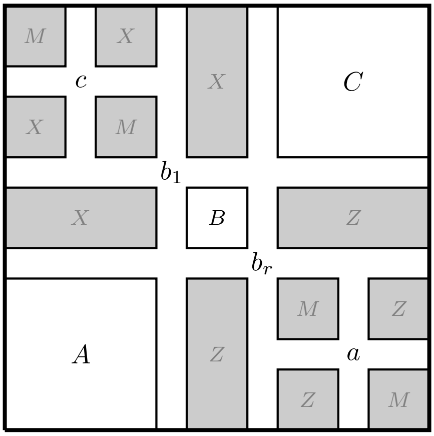


Figure 1: A decomposition

2.3. Class 3

We use the representative triple $\pi_3 := \{3421, 4321, 4312\}$.

Losonczy [6] introduced the notion of maximally clustered elements in a Coxeter group and showed that for Type A (symmetric) groups, they are characterized precisely by avoiding the 3 patterns in π_3. Soon after, Denoncourt and Jones [1] considered heaps in Coxeter groups and found an expression for the generating function for permutations that avoid both π_3 and a heap H as a rational function of the generating function for permutations that avoid 321 and H. The enumeration of π_3-avoiders follows by setting $H = \emptyset$.

For our bijective enumeration, we note that a permutation p avoids π_3 if and only if each of its components does so. So it suffices to determine u_n, the number of indecomposable π_3-avoiders of length n, for then the Invert transform of $(u_n)_{n\geq 1}$ gives the unrestricted counting sequence. Clearly, $u_1 = 1$ and we will show that $u_n = \frac{1}{2}\binom{2(n-1)}{n-1}$ for $n \geq 2$.

The left to right maxima (LR maxima) of a permutation determine a (rotated) Dyck path P with the LR maxima at the NE corners (N = North, E = East), as in Figure 2. The returns to the diagonal split P into its *components*, and P is *indecomposable* if it has exactly one return (necessarily at its endpoint). Components of the permutation p correspond to components of the Dyck path P and so p is indecomposable iff P is.

We begin with an obvious connection between fixed points and 321 patterns.

Lemma 2.5. *For any permutation p and a fixed point b of p, either b is a component of p or b is the "2" of a 321 pattern in p.*

Now we look at the structure of indecomposable π_3-avoiders.

Lemma 2.6. *Let p be an indecomposable π_3-avoider.*

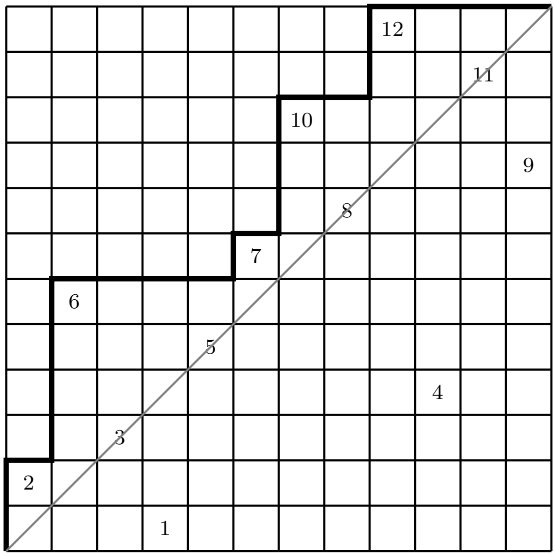


Figure 2: A permutation with LR maxima $2, 6, 7, 10, 12$ and its Dyck path. This permutation is indecomposable

(i) An entry b of p can be the "2" of at most one 321 *pattern.*

(ii) If cba is a 321 *pattern in p, then b is a fixed point of p.*

(iii) A fixed point b is the "2" of exactly one 321 *pattern in p.*

Proof. (i) If b was the "2" of more than one 321 pattern, a forbidden pattern would be present.

(ii) By (i), the entries preceding b are precisely $\{c\} \cup [b-1] \backslash \{a\}$ and so b is the b-th entry.

(iii) This follows from part (i) and Lemma 2.5. ☐

Corollary 2.7. *An indecomposable permutation is a π_3-avoider if and only if it either avoids* 321 *or the "2"s of its* 321 *patterns are all distinct and all fixed points.*

Proof. The "only if" part follows from Lemmas 2.5 and 2.6, and the presence of any one of the offending patterns would imply two 321 patterns with the same 2. ☐

Lemma 2.8. *An indecomposable π_3-avoider is determined by the locations in the matrix diagram of its LR maxima and its fixed points.*

Proof. All other entries must be increasing. Suppose not and that $b > a$ were two other entries, with b to the left of a. Then a LR maximum would precede b, so b would be the "2" of a 321 and hence a fixed point, which it is not. ☐

Arbitrary indecomposable Dyck paths are possible for an indecomposable π_3-avoider, but what about the fixed points? For b to be a fixed point, b cannot be either the value or position of a LR maximum and there must be exactly one LR maximum preceding b and $> b$. In terms of the Dyck path in a matrix diagram, b cannot be in the row or column of a NE corner, and the b-th E step (among the E steps) and its bounce N step must be the end steps of a subpath with just one peak ($= NE$ corner). Any b meeting these conditions can be a fixed point. More precisely, given an indecomposable Dyck path (determining the LR maxima and their positions) and a subset B of the b's meeting the above conditions, there is exactly one indecomposable π_3-avoider with this Dyck path and fixed point set B, namely, the permutation in which all other entries are increasing.

It is convenient to focus on the vertices of the Dyck path, and call a vertex *good* if it is the left endpoint of the E step directly above a possible fixed point b. Since the Dyck path is indecomposable, we may delete the first and last step to get a new (unrestricted) Dyck path of semilength $n-1$ with a new diagonal line joining its endpoints. In this formulation, a vertex is good if (i) it joins 2 E steps, (ii) its *bounce* vertex (down to the diagonal, left to the path) joins 2 N steps, and (iii) the subpath bounded by the vertex and its bounce contains only one peak. Some examples are shown in Figure 3.

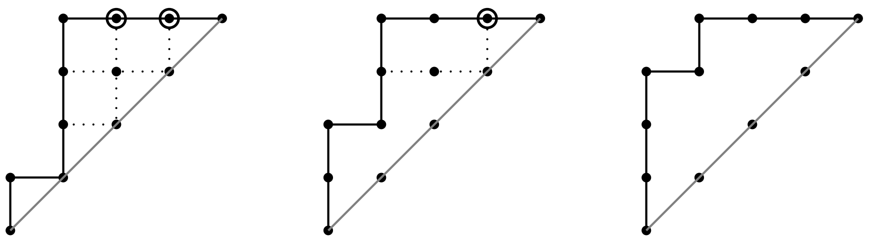


Figure 3: Good vertices

Thus we have shown that indecomposable π_3-avoiders of length n correspond to Dyck paths of semilength $n-1$ in which some (maybe all or none) of the good vertices are marked (with marked vertices corresponding to the fixed points). We now give a bijection from these marked Dyck paths to the set of all balanced paths of $n-1$ N steps and $n-1$ E steps that end with an E step, counted by $\frac{1}{2}\binom{2(n-1)}{n-1}$. For each marked vertex v, draw a line from v down to the diagonal and then, in gray, left to the bounce vertex of v, so the new E steps are colored gray. Erase all lines that can't be "seen" from the diagonal, leaving a new Dyck path with (possibly) some gray steps. Lastly, take each component that ends with a gray step and flip it over the diagonal, and then "forget" the coloring. The result is the desired balanced path. The terminal E step of the Dyck path remains undisturbed and so the balanced path always ends with an E. For example, the permutation in Figure 2 is an indecomposable π_3-avoider of length $n = 12$ with 4 fixed points and it produces the Dyck path of semilength $n - 1 = 11$ with 4 marked vertices in Figure 4a corresponding bijectively to the balanced path in Figure 4b. To reverse

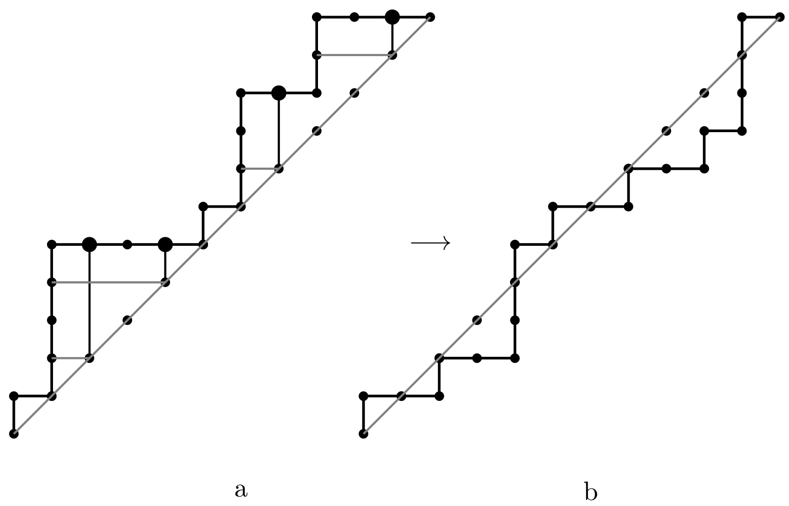


Figure 4: A marked Dyck path (a) and its corresponding balanced path (b)

the map, record the points p on the diagonal that terminate an N step lying below the diagonal. Flip over the diagonal each component that lies below the diagonal. Then, for each p, there is a new E segment (= maximal sequence of contiguous E steps) into p and a N segment out of p that may be new or original. In any case, interchange these E and N segments in the path. Lastly, mark the vertex directly above each p.

2.3.1. Class 3, alternative count

Let a_n be the number of permutations of length n that avoid π_3. In order to study the sequence a_n, we extend our notation by defining $a_n(i_1, i_2, \ldots, i_s)$ to be the number of permutations $\sigma_1\sigma_2\cdots\sigma_n$ of length n that avoid π_3 such that $\sigma_1\sigma_2\cdots\sigma_s = i_1i_2\cdots i_s$.

Lemma 2.9. *We have*

$$a_n(i) = 2a_{n-1}(i) + \sum_{j=1}^{i} a_{n-1}(j) - 2\sum_{j=1}^{i} a_{n-2}(j), \qquad 1 \le i \le n-3,$$

$$a_n(n-2) = 2a_{n-1}(n-2) + \sum_{j=1}^{n-3} a_{n-1}(j) - 2\sum_{j=1}^{n-3} a_{n-2}(j) + a_{n-2} - a_{n-3},$$

with the initial conditions $a_n(n) = a_n(n-1) = a_{n-1}$.

Proof. By the definitions the initial conditions hold, and for $1 \leq i \leq n-2$,

$$a_n(i) = \sum_{j=1}^{i-1} a_n(i,j) + \sum_{j=i+1}^{n-2} a_n(i,j) + a_n(i,n-1) + a_n(i,n).$$

Clearly, $a_n(i,j) = 0$ for all $1 \leq i < j \leq n-2$ and $a_n(i,n-1) = a_n(i,n) = a_{n-1}(i)$ for all $1 \leq i \leq n-2$. Thus,

$$a_n(i) = 2a_{n-1}(i) + \sum_{j=1}^{i-1} a_n(i,j). \tag{2.1}$$

Also, for $1 \leq j < i \leq n-3$,

$$a_n(i,j) = \sum_{\ell=1}^{j-1} a_n(i,j,\ell) + \sum_{\ell=j+1}^{n-1} a_n(i,j,\ell) + a_n(i,j,n) = \sum_{\ell=1}^{j-1} a_{n-1}(j,\ell) + a_{n-1}(i,j),$$

which, by (2.1), implies $a_n(i,j) = a_{n-1}(j) - 2a_{n-2}(j) + a_{n-1}(i,j)$. Hence, (2.1) gives

$$a_n(i) = 2a_{n-1}(i) + \sum_{j=1}^{i} a_{n-1}(j) - 2\sum_{j=1}^{i} a_{n-2}(j), \qquad 1 \leq i \leq n-3,$$

$$a_n(n-2) = 2a_{n-1}(n-2) + \sum_{j=1}^{n-3} a_{n-1}(j) - 2\sum_{j=1}^{n-3} a_{n-2}(j) + a_{n-2} - a_{n-3}.$$

□

In order to solve the recurrence in Lemma 2.9, we define $A_n(v)$ to be the polynomial $\sum_{i=1}^{n} a_n(i)v^{i-1}$. Then, by translating these recurrences in terms of $A_n(v)$, we obtain

$$\begin{aligned} &A_n(v) - a_{n-3}v^{n-3} - a_{n-1}v^{n-1} - a_{n-1}v^{n-2} \\ &= 2(A_{n-1}(v) - a_{n-2}v^{n-2}) + \frac{A_{n-1}(v) - v^{n-2}A_{n-1}(1)}{1-v} \\ &- \frac{2(A_{n-2}(v) - v^{n-2}A_{n-2}(1))}{1-v}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} A_n(v) &= A_{n-3}(1)v^{n-3} + 2A_{n-1}(v) \\ &+ \frac{A_{n-1}(v) - v^{n}A_{n-1}(1)}{1-v} - \frac{2(A_{n-2}(v) - v^{n-1}A_{n-2}(1))}{1-v}, \end{aligned} \tag{2.2}$$

for all $n \geq 3$. By the definitions, we have $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1 + v$.

Now let $A(x, v) = \sum_{n\geq 0} A_n(v)x^n$ be the generating function for the sequence $A_n(v)$. Multiplying (2.2) by x^n, and summing over all $n \geq 3$, yields

$$\begin{aligned} A(x,v) - (1+v)x^2 - x - 1 = x^3A(xv,1) &+ 2x(A(x,v) - x - 1) \\ &+ \frac{x}{1-v}(A(x,v) - 1 - x - v(A(xv,1) - 1 - xv)) \\ &- \frac{2x^2}{1-v}(A(x,v) - 1 - v(A(xv,1) - 1)), \end{aligned}$$

which may be written as

$$\begin{aligned} &\left(1 - 2\frac{x}{v} - \frac{x}{v(1-v)} + \frac{2x^2}{v^2(1-v)}\right) A(x/v, v) \\ &\qquad = \frac{x^2}{v^2} + \frac{x^2}{v} - x - \frac{3x}{v} + 1 + \left(\frac{x^3}{v^3} - \frac{x}{1-v} + \frac{2x^2}{v(1-v)}\right) A(x,1). \end{aligned}$$

To solve the preceding functional equation, we make use of the *kernel method* (see, e.g., [4]). Let $v = v_0(x) = \frac{1+\sqrt{1-4x}}{2}$. Then, the kernel $1 - 2\frac{x}{v} - \frac{x}{v(1-v)} + \frac{2x^2}{v^2(1-v)}$ at $v = v_0(x)$ equals zero, which implies

$$\left(\frac{x}{1-v_0(x)} - \frac{x^3}{v_0^3(x)} - \frac{2x^2}{v_0(x)(1-v_0(x))}\right) A(x,1) = \frac{x^2}{v_0^2(x)} + \frac{x^2}{v_0(x)} - x - \frac{3x}{v_0(x)} + 1.$$

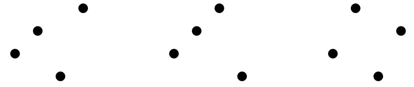
Simplifying the formula found for $A(x, 1)$ yields, after several algebraic steps, the following result.

Theorem 2.10. *The generating function for the number of permutations of length n that avoid π_3 is given by*

$$\frac{2(1-4x)}{2 - 9x + 4x^2 - x\sqrt{1-4x}}.$$

2.4. Class 4

$\pi_4 = \{2314, 2341, 2413\}$



Let $A_n = S_n(\pi_4)$. Let $\sigma \in A_n$ with $n \geq 2$. By considering the positions of $n-1$ and n within σ, one can show the following block decomposition result.

Lemma 2.11. *Let $n \geq 2$. A permutation σ of length n avoids π_4 if and only if either*

- $\sigma = \sigma'(n-1)\sigma''n\sigma'''$ *such that* $\sigma' > \sigma''\sigma'''$ *(that is, each letter of* σ' *is greater than each letter of* σ'' *or* σ'''*),* σ' *is a permutation of* $[n-j+1, n-2]$ *that avoids* 231*, and* $\sigma''n\sigma'''$ *is a permutation of* $\{1, 2, \ldots, n-j, n\}$ *that avoids* π_4*; or*

- $\sigma = \sigma'n\sigma''n-1\sigma'''$*: If* $\sigma' = \emptyset$*, then* $\sigma \in A_n$ *if and only if* $\sigma''(n-1)\sigma''' \in A_{n-1}$*. If* $\sigma' \neq \emptyset$ *and* $\sigma'' = \emptyset$*, then* $\sigma \in A_n$ *if and only if* $\sigma'(n-1)\sigma''' \in A_{n-1}$*. If* $\sigma', \sigma'' \neq \emptyset$*, then* $\sigma' > \sigma''\sigma'''$*,* σ' *avoids* 231*, and* $\sigma''(n-1)\sigma'''$ *avoids* π_4*.*

Define $A(x) = \sum_{n\geq 0} \#A_n x^n$. Since 231-avoiders are counted by Catalan numbers, we have by Lemma 2.11,

$$\begin{aligned} A(x) &= 1 + x + xC(x)(A(x)-1) \\ &\quad + x(A(x)-1) + x(A(x)-1-xA(x)) + x(C(x)-1)(A(x)-1-xA(x)), \end{aligned}$$

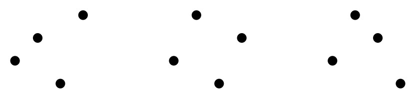
where $A(x) - 1 - xA(x)$ is the generating function for the number of permutations $\sigma_1 \cdots \sigma_n$ in A_n, $n \geq 2$, with $\sigma_1 \neq n$. Thus, we can state the following result.

Theorem 2.12. *The generating function for the number of permutations of length* n *that avoid* π_4 *is given by*

$$\frac{2(1-4x)}{2 - 9x + 4x^2 - x\sqrt{1-4x}}.$$

2.5. Class 5

$\pi_5 = \{2314, 2413, 2431\}$



Let $A_n = S_n(\pi_5)$. Let $\sigma \in A_n$ with $n \geq 2$. Again, we have a block decomposition of σ.

Lemma 2.13. *Let* $n \geq 2$*. A permutation* σ *of length* n *avoids* π_5 *if and only if either*

- $\sigma = \sigma'n\sigma''(n-1)\sigma'''$ *such that* $\sigma''(n-1)\sigma''' > \sigma'$*,* σ' *is a permutation of* $[j]$ *that avoids* 231*, and* $\sigma''(n-1)\sigma'''$ *is a permutation of* $[j+1, n-1]$ *that avoids* π_5*; or*

- $\sigma = \sigma'(n-1)\sigma''n\sigma'''$*: If* $\sigma' = \emptyset$*, then* $\sigma \in A_n$ *if and only if* $\sigma''(n-1)\sigma''' \in A_{n-1}$*. If* $\sigma' \neq \emptyset$ *and* $\sigma'' = \emptyset$*, then* $\sigma \in A_n$ *if and only if* $\sigma'(n-1)\sigma''' \in A_{n-1}$*. If* $\sigma', \sigma'' \neq \emptyset$*, then* $\sigma''n\sigma''' > \sigma'$*,* σ' *is a permutation of* $[j]$ *that avoids* 231*, and* $\sigma''(n-1)\sigma'''$ *is a permutation of* $\{j+1, j+2, \ldots, n-2, n\}$ *that avoids* π_5*.*

Define $A(x) = \sum_{n\geq 0} \#A_n x^n$. By Lemma 2.13, we have

$$A(x) = 1 + x + xC(x)(A(x) - 1) \\ + x(A(x) - 1) + x(A(x) - 1 - xA(x)) + x(C(x) - 1)(A(x) - 1 - xA(x)),$$

where $A(x) - 1 - xA(x)$ is the generating function for the number of permutations $\sigma_1 \cdots \sigma_n$ in A_n, $n \geq 2$, with $\sigma_1 \neq n$. Thus, we can state the following result.

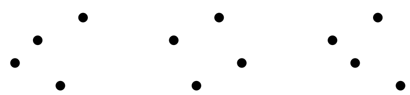
Theorem 2.14. *The generating function for the number of permutations of length n that avoid π_5 is given by*

$$\frac{2(1-4x)}{2 - 9x + 4x^2 - x\sqrt{1-4x}}.$$

Note that Lemmas 2.11 and 2.13 yield a recursive bijection between $S_n(\pi_4)$ and $S_n(\pi_5)$.

2.6. Class 6

We use the representative triple $\pi_6 = \{2314, 3142, 3241\}$



In order to determine the number of π_6-avoiders of length n, we refine the set by considering a couple of auxiliary statistics as follows. Given $n \geq 2$, $\ell \in [n-1]$, and $1 \leq i \leq \ell$, let $u(n;\ell,i)$ denote the number of permutations of length n avoiding the patterns in π_6 in which the largest letter (if it exists) to the left of n is ℓ wherein there are exactly $i-1$ positions separating ℓ and n. Let $u(n;\ell) := \sum_{i=1}^{\ell} u(n;\ell,i)$. Denote by $u(n)$ the number of permutations of length n avoiding the patterns in π_6, the set of which we will denote by $\mathcal{U}_n$. Since members of $\mathcal{U}_n$ starting with n are synonymous with members of $\mathcal{U}_{n-1}$, we have the relation

$$u(n) = u(n-1) + \sum_{\ell=1}^{n-1} u(n;\ell), \qquad n \geq 2, \tag{2.3}$$

with $u(1) = u(0) = 1$. The following lemma provides a recurrence for the array $u(n;\ell,i)$ which we will use to determine $u(n)$.

Lemma 2.15. *If $2 \leq i \leq \ell < n$, then*

$$u(n;\ell,i) = C_{\ell-i}C_{i-1}u(n-\ell-1), \qquad i \geq 2, \tag{2.4}$$

with

$$u(n;\ell,1) = C_{n-\ell-1}u(\ell) + C_{\ell-1}u(n-\ell-1) - C_{\ell-1}C_{n-\ell-1}$$

$$+ (C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1), \tag{2.5}$$

for $\ell \geq 2$, *and* $u(n;1,1) = u(n-2)$ *for* $n \geq 2$.

Proof. That $u(n;1,1) = u(n-2)$ for $n \geq 2$ follows from the definitions. We give combinatorial proofs of (2.4) and (2.5). Let $\mathcal{U}_{n,\ell,i}$ denote the subset of $\mathcal{U}_n$ enumerated by $u(n;\ell,i)$. To show (2.4), note first that members of $\mathcal{U}_{n,\ell,i}$, where $2 \leq i \leq \ell$, must be of the form $\alpha = \alpha_1 \ell \alpha_2 n \delta$, where α_2 has length $i-1$ and δ comprises the elements of $[\ell+1, n-1]$. (Note α_2 non-empty implies that there can be no members of $[\ell-1]$ to the right of n, for otherwise there would be an occurrence of 3241 or 3142 in which the roles of the "3" and "4" are played by the ℓ and n, respectively.) Furthermore, any letter in α_1 must be smaller than any letter in α_2 in order to avoid 2314. Finally, the subwords α_1 and α_2 must both avoid 231 (since n lies to their right), while there is no further restriction on δ (i.e., it must only avoid the original patterns in π_6). Conversely, any permutation α of $[n]$ of the form described above in which α_1 and α_2 both avoid 231, each letter of α_2 is greater than each letter of α_1, and δ avoids the patterns in π_6 is seen to be a member of $\mathcal{U}_{n,\ell,i}$. This implies $u(n;\ell,i) = C_{\ell-i}C_{i-1}u(n-\ell-1)$ for $2 \leq i \leq \ell$, as desired.

To show (2.5), let $X = \mathcal{U}_{n,\ell,1}$ and first consider the case in which there are no elements of $[\ell-1]$ occurring to the right of the letter n within a member of X. Then such members of X may be decomposed as $\alpha \ell n \beta$, where α is a permutation of $[\ell-1]$ avoiding the pattern 231 and $\beta \in \mathcal{U}_{n-\ell-1}$ (on the letters in $[\ell+1, n-1]$). Furthermore, permutations of this form are seen to avoid the patterns in π_6. Thus, there are $C_{\ell-1}u(n-\ell-1)$ possibilities in this case.

Now suppose that all elements of $[\ell-1]$ occur to the right of n within $\rho \in X$. We consider subcases as follows. First assume ρ is of the form $\rho = \ell n \rho_1 \rho_2$, where ρ_1 and ρ_2 are permutations of $[\ell+1, n-1]$ and $[\ell-1]$, respectively. Then ρ_1 must avoid the pattern 213 since $\rho_2 \neq \emptyset$, while ρ_2 has no restrictions other than those imposed by π_6. This implies that there are $C_{n-\ell-1}u(\ell-1)$ possibilities in this case. Now assume that at least one letter of $[\ell-1]$ lies to the left of some letter of $[\ell+1, n-1]$ within ρ. Then ρ must be of the form $\rho = \ell n \delta_1 \gamma \delta_2$ in this case, where γ consists of all the letters in $[\ell-1]$ and δ_1 and δ_2 together comprise all of the letters in $[\ell+1, n-1]$, with δ_2 non-empty. (For otherwise, there would be a guaranteed occurrence of 3241 or 3142, with the ℓ playing the role of the "3".) Furthermore, since $\ell \geq 2$ implies γ is non-empty, it must be the case that all letters of δ_1 are larger than all letters of δ_2 in order to avoid 2314. In addition, γ non-empty implies δ_1 must avoid 213 and δ_2 non-empty implies γ must avoid 231. Finally, the subword δ_2 is seen to have no restrictions other than those imposed by π_6 since all letters of δ_1 and γ are larger or smaller, respectively, than all letters of δ_2. Since the preceding conditions on γ, δ_1 and δ_2 are seen also to be sufficient for membership of ρ within X, it follows that there are $C_{\ell-1}\sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1)$ possibilities in this case, where r denotes the length of δ_1.

Now suppose that there is at least one element of $[\ell-1]$ to the left and to the right of n within $\beta \in X$, whence $\ell \geq 3$ in this case. Then β can be expressed in the form $\beta = \beta_1 \ell n \delta_1 \beta_2 \delta_2$, where β_1, β_2 are non-empty words in $[\ell - 1]$ and δ_1, δ_2 are words in $[\ell + 1, n - 1]$. First assume δ_2 is non-empty. Note that all elements of β_1 must be less than all of those in β_2 in this case in order to avoid 2314 (for otherwise, there would be an occurrence of 2314 in which the ℓ plays the role of the "3" and any member of δ_2 plays the role of the "4"). Let p be the smallest element of β_2. Then $2 \leq p \leq \ell - 1$ since both β_1 and β_2 are non-empty. Furthermore, δ_2 non-empty implies both β_1 and β_2 avoid 231, which implies that there are $\sum_{p=2}^{\ell-1} C_{p-1}C_{\ell-p} = C_\ell - 2C_{\ell-1}$ possibilities for β_1 and β_2. Once the positions of the letters in β_1 and β_2 have been determined, there are $\sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1)$ possibilities for the letters in δ_1 and δ_2, upon considering the length r of δ_1 (note that all letters in δ_2 must be smaller than all letters in δ_1 in order to avoid 2314). Thus, there are $(C_\ell - 2C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n - \ell - r - 1)$ members β of the form above in which $\delta_2 \neq \emptyset$.

Finally, suppose $\delta_2 = \emptyset$ in the decomposition of β above. In this case, the subsequence $\beta_1 \ell \beta_2$ constitutes a permutation of $[\ell]$ avoiding the patterns in π_6 which does not start or end with the letter ℓ. By subtraction, there are $u(\ell) - u(\ell - 1) - C_{\ell-1}$ possibilities for this subsequence. The letters of δ_1 must avoid 213, with no other restrictions on δ_1. Furthermore, any permutation β of the form above satisfying the stated conditions on its constituent parts is seen to avoid the patterns in π_6. Since there are $C_{n-\ell-1}$ possibilities for δ_1, it follows that there are $(u(\ell) - u(\ell-1) - C_{\ell-1})C_{n-\ell-1}$ permutations β of the form above in which $\delta_2 = \emptyset$. Combining all of the previous cases implies that the cardinality of X is given for $\ell \geq 2$ by

$$\begin{aligned}
&C_{\ell-1}u(n-\ell-1) + C_{n-\ell-1}u(\ell-1) + C_{\ell-1} \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1) \\
&+ (C_\ell - 2C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1) + (u(\ell) - u(\ell-1) - C_{\ell-1})C_{n-\ell-1} \\
&= C_{n-\ell-1}u(\ell) + C_{\ell-1}u(n-\ell-1) - C_{\ell-1}C_{n-\ell-1} \\
&+ (C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1),
\end{aligned}$$

which gives (2.5). □

Summing (2.4) over $2 \leq i \leq \ell$, and using the recurrence for Catalan numbers, implies

$$u(n;\ell) = u(n;\ell,1) + (C_\ell - C_{\ell-1})u(n-\ell-1), \qquad \ell \geq 1. \tag{2.6}$$

Summing (2.6) over $1 \leq \ell \leq n-1$, and using (2.5), implies

$$\sum_{\ell=1}^{n-1} u(n;\ell) = \sum_{\ell=1}^{n-1} u(n;\ell,1) + \sum_{\ell=1}^{n-1} (C_\ell - C_{\ell-1})u(n-\ell-1)$$

$$= u(n-2) + \sum_{\ell=2}^{n-1} C_{n-\ell-1}u(\ell) + \sum_{\ell=2}^{n-1} C_{\ell-1}u(n-\ell-1) - \sum_{\ell=2}^{n-1} C_{\ell-1}C_{n-\ell-1}$$
$$+ \sum_{\ell=2}^{n-1}(C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1) + \sum_{\ell=1}^{n-1}(C_\ell - C_{\ell-1})u(n-\ell-1).$$

Thus, we have by (2.3),

$$\begin{aligned} u(n) &= \sum_{\ell=0}^{n-1} C_{n-\ell-1}u(\ell) - C_{n-1} + \sum_{\ell=1}^{n-1} C_{\ell-1}u(n-\ell-1) - \sum_{\ell=1}^{n-1} C_{\ell-1}C_{n-\ell-1} \\ &+ \sum_{\ell=1}^{n-1}(C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1) + \sum_{\ell=0}^{n-1}(C_\ell - C_{\ell-1})u(n-\ell-1) \\ &= 2\sum_{\ell=1}^{n-1} C_{n-\ell-1}u(\ell) + C_{n-1} - \sum_{\ell=1}^{n-1} C_{\ell-1}C_{n-\ell-1} \\ &+ \sum_{\ell=1}^{n-1}(C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1) \\ &= 2\sum_{\ell=1}^{n-1} C_{n-\ell-1}u(\ell) + \sum_{\ell=1}^{n-1}(C_\ell - C_{\ell-1}) \sum_{r=0}^{n-\ell-2} C_r u(n-\ell-r-1), \qquad n \geq 2. \end{aligned} \tag{2.7}$$

Let $f(x) = \sum_{n\geq 1} u(n)x^n$. Multiplying both sides of (2.7) by x^n, and summing over $n \geq 2$, yields

$$\begin{aligned} f(x) &= x + 2xC(x)f(x) + \sum_{\ell\geq 1}(C_\ell - C_{\ell-1}) \sum_{r\geq 0} C_r \sum_{n\geq \ell+r+2} u(n-\ell-r-1)x^n \\ &= x + 2xC(x)f(x) + x\sum_{\ell\geq 1}(C_\ell - C_{\ell-1})x^\ell \sum_{r\geq 0} C_r x^r \sum_{n\geq 1} u(n)x^n \\ &= x + 2xC(x)f(x) + x((1-x)C(x)-1)C(x)f(x) \\ &= x + xC(x)f(x) + (1-x)(C(x)-1)f(x) \\ &= x + (C(x)+x-1)f(x), \end{aligned}$$

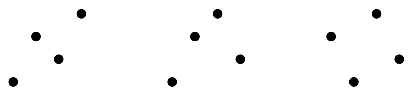
where we have used the fact $xC^2(x) = C(x) - 1$. Thus, we have

$$\begin{aligned} \sum_{n\geq 0} u(n)x^n = 1 + f(x) = \frac{2 - C(x)}{2 - x - C(x)} &= \frac{1 - 4x - \sqrt{1-4x}}{1 - 4x + 2x^2 - \sqrt{1-4x}} \\ &= \frac{2(1-4x)}{2 - 9x + 4x^2 - x\sqrt{1-4x}}, \end{aligned}$$

as desired.

2.7. Class 7

$\pi_7 = \{1324, 1342, 3142\}$



To count π_7-avoiders by first entry m, set $u(n) = |S_n(\pi_7)|$ and $u(n,m) = |\{p \in S_n(\pi_7) : p_1 = m\}$.

Clearly, $u(n,m) = u(n-1)$ for $m = n$. For $1 \le m < n$, use the left to right maxima $(m_i)_{i=1}^{k+1}$, where $k \ge 1,\ m_1 = m$, and $m_{k+1} = n$, to decompose p as

$$p = m_1 A_1 m_2 A_2 \cdots m_k A_k m_{k+1} A_{k+1}. \tag{2.8}$$

Proposition 2.16.
(*i*) $m_1, \ldots, m_k$ *are consecutive integers.*
(*ii*) $A_1 > A_2 > \cdots > A_k > [m-1] \cap A_{k+1}$, *where* $A_i > A_j$ *means* $min(A_i) > max(A_j)$.
(*iii*) *For* $1 \le i \le k,\ A_i$ *avoids* 132.

Proof. (i) Say $m_i = a$ and $m_{i+1} = c \ge a+2$. Then $b := a+1$ occurs after m_{i+1} and $\{a, c, b, n\}$ occur either in the order $acbn$ (1324) or $acnb$ (1342), both forbidden.

(ii) If $a_i < a_j$ with $1 \le i < j \le k+1,\ a_i \in A_i,\ a_j \in A_j$, then $m_i a_i m_j a_j$ is a 3142.

(iii) If not, then $n = m_{k+1}$ would be the "4" of a 1324. ☐

Thus p is captured by the list (recall St refers to standardizing a list)

$$\mathrm{St}(A_1), \ldots, \mathrm{St}(A_k), \mathrm{St}(m_1 A_{k+1})\,.$$

Conversely, if these conditions hold and $\mathrm{St}(m_1 A_{k+1})$ is a π_7-avoider, then so is p.

Since 132-avoiders of length n are equinumerous with Dyck paths of size (semi-length) n, and $(k+1)$-lists of Dyck paths of total size n are counted by the generalized Catalan number $C(n,k) := (k+1)\binom{2n+k+1}{n}/(2n+k+1)$, the decomposition (2.8) leads to the recurrence

$$u(n,m) = \sum_{k=1}^{n-m} \sum_{h=0}^{m-1} C(m-h-1, k-1)\, u(n-m+h-k+1, h+1),$$

where the index h refers to the number of entries of A_{k+1} that are $< m_1$. Recall that the generating function $C(x,y) := \sum_{n,k \ge 0} C(n,k) x^n y^k$ is given by $C(x,y) = C(x)/(1-yC(x))$ where $C(x)$ is the generating function for the Catalan numbers.

Now define generating functions $F(x) = \sum_{n\ge1} u(n)x^n$ and

$$F(x,y) = \sum_{n\ge1} \sum_{m=1}^{n} u(n,m) x^n y^m.$$

Note that $F(x) = F(x,1)$.

Split $F(x,y)$ into $F_1 + F_2$, where F_1 is the sum over $m < n$ and F_2 is the sum over $m = n$. Using the recurrence, we have

$$F_1 = \sum_{n\geq 2}\sum_{m=1}^{n-1}\sum_{k=1}^{n-m}\sum_{h=0}^{m-1} C(m-h-1,k-1)u(n-m+h-k+1,h+1)x^n y^m.$$

Introduce new summation indices $r = m-h-1, s = k-1, t = n-m+h-k+1, j = h+1$ to get

$$F_1 = \sum_{r,s\geq 0, t\geq 1}\sum_{j=1}^{t} C(r,s)u(t,j)x^{r+s+t+1}y^{j+r} = xC(xy,x)F(x,y)\,.$$

Also, we have

$$F_2 = \sum_{n\geq 1} u(n-1)(xy)^n = xy\big(1+F(xy,1)\big)\,.$$

So $F(x,y)$ satisfies

$$F(x,y) = xC(xy,x)F(x,y) + xy + xyF(xy,1)\,. \tag{2.9}$$

Set $y = 1$ in (2.9) to get $F(x,1) = x/(1-x-xC(x,x))$, leading to

$$F(x) = \frac{x}{1-x-\frac{xC(x)}{1-xC(x)}},$$

and, after expansion,

$$F(x,y) = \frac{xy\big(1+F(xy)\big)}{1-xC(xy,x)}\,,$$

and $1+F(x) = \frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}$.

As an aside, the decomposition (2.8) readily yields a bijection from $S_n(\pi_7)$ to a certain subset of the Schroder paths of size $n-1$. We represent a Schroder path as a Motzkin path consisting of upsteps $U = (1,1)$, flatsteps $F = (1,0)$ and downsteps $D = (1,-1)$, but with size measured by # U's + # F's rather than length. Let $\mathcal{A}_n$ denote the set of Schroder paths of size n with all flatsteps at ground level, ending with an F, and decorated so that, for each descent (maximal sequence of contiguous downsteps) that ends at ground level, one of its downsteps is marked. Let $\mathcal{B}_n$ denote the set of Schroder paths of size n such that, for each flatstep not at ground level, the portion of the path between the flatstep and the next vertex at ground level consists of a Dyck path (possibly empty) followed only by downsteps. There is a simple bijection from $\mathcal{A}_n$ to $\mathcal{B}_{n-1}$: delete the last step (necessarily F) and, for each marked downstep, if it is the last downstep of a descent, just erase the mark, otherwise delete the marked step and turn its matching upstep into a flatstep. For example, here is a bijection from $S_n(\pi_7)$ to the paths in $\mathcal{A}_n$. Let ϕ be your favorite bijection from 312-avoiders to Dyck paths. Given $p \in S_n(\pi_7)$, if the first

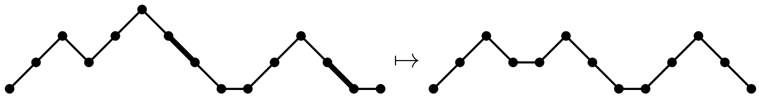


Figure 5: The bijection $\mathcal{A}_n \longrightarrow \mathcal{B}_{n-1}$

entry of p is n, begin the path with a flatstep, delete n, and start over. Otherwise, consider the decomposition (2.8). Replace each $m_i, 1 \le i \le k$, by an upstep, each $A_i, 1 \le i \le k$, by the Dyck path $\phi\big(\mathrm{St}(A_i)\big)$, append k downsteps and mark the first one. These replacements and the appendage produce a primitive Dyck path with one marked downstep on the last descent. Next, ignore the entry $m_{k+1} = n$ and start over with $\mathrm{St}(m_1 A_{k+1})$. The process will end when $\mathrm{St}(m_1 A_{k+1}) = 1$, which will terminate the path with a flatstep.

2.7.1. Class 7, alternative count

Let $b(n; i, j)$ denote the number of permutations of length n avoiding the patterns in π_7 in which the first letter is i and the second is j. If $n \ge 2$, then define $b(n; i) = \sum_{j=1, j\neq i}^{n} b(n; i, j)$ and $b(n) = \sum_{i=1}^{n} b(n; i)$, with $b(1) = b(1; 1) = 1$. Put $b(n; i, j) = 0$ if $i = 0$ or $j = 0$.

We have the following obvious initial values. If $n = 2$, then $b(2) = 2$, with $b(2; 1) = b(2; 1, 2) = 1$ and $b(2; 2) = b(2; 2, 1) = 1$. If $n = 3$, then $b(3) = 6$, with $b(3; 1) = b(3; 2) = b(3; 3) = 2$ and $b(3; 1, 2) = b(3; 1, 3) = b(3; 2, 1) = b(3; 2, 3) = b(3; 3, 1) = b(3; 3, 2) = 1$.

If $n \ge 4$, then the array $b(n; i, j)$ is determined as follows.

Lemma 2.17. *If $1 \le i \le n-1$, then $b(n; i, i+1) = b(n; i, n) = b(n; n, i) = b(n-1; i)$, with $b(n; i, i-1) = b(n-1; i-1)$ for $1 < i \le n$. If $1 \le i < j-1 < n-1$, then $b(n; i, j) = 0$. If $1 \le j < i-1 < n-1$, then*

$$b(n; i, j) = b(n-1; i-1, j) + \sum_{k=1}^{j-1} b(n-1; i-1, k). \tag{2.10}$$

Proof. Let $\mathcal{B}_n$ denote the subset of the permutations of length n avoiding the patterns in π_7 and $\mathcal{B}_{n,i,j}$ the subset of $\mathcal{B}_n$ enumerated by $b(n; i, j)$. The first statement is clear since a letter n in either the first or second position is seen to be extraneous concerning the avoidance of the patterns in π_7, as is the letter $i+1$ within members of $\mathcal{B}_{n,i,i+1}$ and the letter $i-1$ within members of $\mathcal{B}_{n,i,i-1}$. Permutations of length at least four starting with the letters i, j where $1 \le i < j-1 < n-1$ always contain an occurrence of either 1324 or 1342, which implies $b(n; i, j) = 0$ in these cases.

To show (2.10), we consider the third letter k within a member of $\mathcal{B}_{n,i,j}$ where $1 \le j < i-1 < n-1$. Note that k cannot belong to $[i+1, n]$, for if it did, then there would be an occurrence of 3142, as witnessed by any subsequence $ijkx$, where $x \in [j+1, i-1]$. It also cannot be the case that k belongs to $[j+2, i-1]$, for otherwise there would be an occurrence of 1342 or 1324 with either $jkn(j+1)$ or $jk(j+1)n$. Thus, it must be the case that $k = j+1$ or $k \in [j-1]$. The first term on the right-hand side of (2.10) accounts for when $k = j+1$ since the letter k is seen to be extraneous in this case concerning the avoidance of the patterns in π_7 and thus may be deleted.

So assume $k \leq j-1$, and we will show that the letter j may be deleted from members of $\mathcal{B}_{n,i,j}$ in this case. Given $\lambda \in \mathcal{B}_{n-1,i-1,k}$, let λ' be obtained from λ by inserting j between the $i-1$ and k and increasing all letters of λ in $[j, n-1]$ by one. We will show that if λ avoids the patterns in π_7, then so must λ'. Suppose, to the contrary, that λ' contains an occurrence of some pattern $\rho \in \pi_7$. Then ρ cannot be either 1342 or 1324, for otherwise the letter j would play the role of the "1" in an occurrence of either pattern within λ', and replacing j with $k < j$ would imply λ contains one of these patterns, a contradiction. Thus ρ must be 3142. Note that the role of the "3" must be played by the letter j, for otherwise λ would contain an occurrence of 3142 with the "3" and "1" played by $i-1$ and k, respectively.

Thus, the occurrence of 3142 in λ' is realized by a subsequence $j\ell rs$. Note that $r < i$, for otherwise λ would contain an occurrence of 3142 with $(i-1)\ell(r-1)s$, which is impossible. We now consider the position of the element n within λ'. If n lies to the left of r within λ', then $(i-1)k(n-1)(r-1)$ would form an occurrence of 3142 in λ, a contradiction. On the other hand, if n lies to the right of r within λ', then there would be an occurrence of 1324 or 1342 within λ' as witnessed by either ℓrsn or ℓrns, a contradiction. Thus, λ' must avoid the patterns in π_7 if λ does, which completes the proof. □

Define $b(n;i|w) = \sum_{j=1}^{n} b(n;i,j)w^{j-1}$ and

$$B_n(v,w) = \sum_{i=1}^{n}\sum_{j=1}^{n} b(n;i,j)v^{i-1}w^{j-1}.$$

Then the recurrence (2.10) implies

$$\begin{aligned}
&b(n;i|w) - b(n-1;i-1)w^{i-2} - b(n-1;i)w^i\delta_{i<n-1} - b(n-1;i)w^{n-1}\\
&= \sum_{j=1}^{i-2} w^{j-1}\sum_{k=1}^{j} b(n-1;i-1,k)\\
&= \frac{1}{1-w}\bigg(b(n-1;i-1|w) - w^{i-2}b(n-1;i-1|1)\\
&\qquad + b(n-2;i-1)((1+\delta_{i<n-1})w^{i-2} - w^{i-1} - w^{n-2}\delta_{i<n-1})\bigg),
\end{aligned}$$

which implies

$$\begin{aligned}
b(n;i|w) &= b(n-1;i-1)w^{i-2} + b(n-1;i)w^i\delta_{i<n-1} + b(n-1;i)w^{n-1}\\
&\quad + b(n-2;i-1)w^{i-2} + \frac{1}{1-w}(b(n-1;i-1|w) - w^{i-2}b(n-1;i-1|1)
\end{aligned}$$

$$+ b(n-2;i-1)(w^{i-2}-w^{n-2})\delta_{i<n-1}). \tag{2.11}$$

Note that $b(n;1|w) = 2^{n-3}(w+w^{n-1})$ and $b(n;n|w) = \sum_{j=1}^{n-1} b(n-1;j)w^{j-1} = B_{n-1}(w,1)$. Also, $b(n;2,j)$ equals 2^{n-3}, 0, or $b(n-1;2)$ when $j=1$, $4 \le j \le n-1$, or $j=3,n$, respectively. Thus, $b(n;2|w) = 2^{n-3} + b(n-1;2|1)(w^2+w^{n-1})$, which, by induction, implies $b(n;2|w) = 2^{n-3} + (n-2)2^{n-4}(w^2+w^{n-1})$.

Multiplying (2.11) by v^{i-1}, and summing over $i = 3,4,\ldots,n-1$, implies

$$\begin{aligned}
B_n(v,w) &= B_{n-1}(w,1)v^{n-1} + (v+w)B_{n-1}(vw,1) - (vw)^{n-2}(v+w)B_{n-2}(1,1) \\
&\quad + w^{n-1}B_{n-1}(v,1) + vB_{n-2}(vw,1) - v^{n-2}w^{n-3}B_{n-3}(1,1) \\
&\quad + \frac{v}{1-w}\Big(B_{n-1}(v,w) - v^{n-2}B_{n-2}(w,1) - B_{n-1}(vw,1) \\
&\qquad + (vw)^{n-2}B_{n-2}(1,1) + B_{n-2}(vw,1) - w^{n-2}B_{n-2}(v,1)\Big),
\end{aligned}$$

with $B_0(v,w) = B_1(v,w) = 1$, $B_2(v,w) = v+w$ and $B_3(v,w) = v+v^2+w+w^2+vw^2+wv^2$.

Define $B(x;v,w) = \sum_{n\ge 0} B_n(v,w)x^n$. Multiplying the last recurrence by x^n and summing over $n \ge 4$, we obtain after several algebraic steps

$$\begin{aligned}
\frac{1-vx-w}{1-w}B(x;v,w) &= 1-(v+w+1)x - vx^2 \\
&\quad - \frac{x(vwx+vw-2vx-w+w^2)}{1-w}B(x;vw,1) \\
&\quad + \frac{x(1-vx-w)}{1-w}(B(vx,w,1)+B(wx;v,1)) \\
&\quad + \frac{x^2(vw+wvx+w^2-w-vx)}{1-w}B(vwx;1,1).
\end{aligned}$$

Substituting $w = 1-vx$ into the preceding functional equation yields

$$1 = (2+v)x + (1-vx-2x)B(x;v(1-vx),1) - x(1-vx-x)B(vx(1-vx);1,1).$$

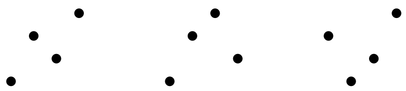
Let v be a solution of the equality $v(1-vx)=1$, namely, $v = C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Replacing v by $C(x)$ in the last functional equation then gives

$$B(x;1,1) = \frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}},$$

as desired.

2.8. Class 8

$\pi_8 = \{1324, 1342, 3124\}$



Let $a(n;i,j)$ denote the number of permutations of length n avoiding the patterns in π_8 in which the first letter is i and the second is j. If $n \geq 2$, then define $a(n;i) = \sum_{j=1,j\neq i}^{n} a(n;i,j)$ and $a(n) = \sum_{i=1}^{n} a(n;i)$, with $a(1) = a(1;1) = 1$. Put $a(n;i,j) = 0$ if $i = 0$ or $j = 0$.

We have the following obvious initial values. If $n = 2$, then $a(2) = 2$, with $a(2;1) = a(2;1,2) = 1$ and $a(2;2) = a(2;2,1) = 1$. If $n = 3$, then $a(3) = 6$, with $a(3;1) = a(3;2) = a(3;3) = 2$ and $a(3;1,2) = a(3;1,3) = a(3;2,1) = a(3;2,3) = a(3;3,1) = a(3;3,2) = 1$.

If $n \geq 4$, then the array $a(n;i,j)$ is determined as follows.

Lemma 2.18. *If $1 \leq i \leq n-1$, then $a(n;i,i+1) = a(n;i,n) = a(n;n,i) = a(n-1;i)$, with $a(n;i,i-1) = a(n-1;i-1)$ for $1 < i \leq n$. If $1 \leq i < j-1 < n-1$, then $a(n;i,j) = 0$. If $1 \leq j < i-1 < n-1$, then*

$$a(n;i,j) = a(n-1;i,j) + a(n-1;i-1,j-1) + \sum_{k=1}^{j-2} a(n-1;j,k). \qquad (2.12)$$

Proof. Let $\mathcal{A}_n = S_n(\pi_8)$ and $\mathcal{A}_{n,i,j}$ be the subset of $\mathcal{A}_n$ enumerated by $a(n;i,j)$. The first statement is clear since a letter n in either the first or second position is seen to be extraneous concerning the avoidance of the patterns in π_8, as is the letter $i+1$ within members of $\mathcal{A}_{n,i,i+1}$ and the letter $i-1$ within members of $\mathcal{A}_{n,i,i-1}$. Permutations of length at least four starting with the letters i, j where $1 \leq i < j-1 < n-1$ must contain an occurrence of either 1324 or 1342, whence $a(n;i,j) = 0$ in these cases.

We now show (2.12). To do so, we consider the third letter k within a member of $\mathcal{A}_{n,i,j}$ where $1 \leq j < i-1 < n-1$. Note that k cannot belong to $[i+1, n-1]$, for if it did, then there would be an occurrence of 1342 or 1324, as witnessed by either $jkn(i-1)$ or $jk(i-1)n$. It also cannot belong to $[j+1, i-1]$, for if it did, then there would be an occurrence of 3124, as witnessed by $ijkn$. Thus, it must be the case that $k = n$ or $k \in [j-1]$. It is seen that the first two terms on the right-hand side of (2.12) account for the cases in which $k = n$ or $k = j-1$, respectively. Now assume $k \in [j-2]$. In this case, we will argue that the letter i is superfluous when considering the avoidance of patterns in π_8, whence it may be deleted. This will give the sum on the right-hand side of (2.12) and complete the proof. Given $\lambda \in \mathcal{A}_{n-1,j,k}$, let λ' be obtained from λ by writing the letter i before λ and increasing all elements of $[i, n-1]$ within λ by one. We will show that if λ avoids the patterns in π_8, then so does λ'. Suppose, to the contrary, that λ' contains an occurrence of some pattern ρ of π_8. Since λ avoids the patterns in π_8, we must have $\rho = 3124$, with the letter i playing the role of the "3".

Suppose that the 3124 subsequence within λ' is witnessed by $i\ell rs$. Note that $r > j$, for otherwise λ would contain an occurrence of 3124 with the subsequence

$j\ell r(s-1)$. We consider several cases on ℓ. First assume $\ell \in [j+1, i-1]$. Then all elements of $[k+1, j-1]$ within λ' must occur to the left of r in order to avoid 1342, and thus to the left of ℓ as well in order to avoid 1324. But then λ would contain 3124 as witnessed by $jkx\ell$, where x is any element of $[k+1, j-1]$, a contradiction. On the other hand, if $\ell \in [k+1, j-1]$, then λ would contain 3124 with the subsequence $jk\ell r$, which is again not possible. Finally, let us assume $\ell \in [k]$; note that $\ell = j$ is included in this case, for if the second letter in an occurrence of 3124 starting with i is j, then one may replace j with k since $k < j$. Note that then any $x \in [k+1, j-1]$ must lie to the left of s within λ', for if x was to the right of s, then $kr(s-1)x$ would be an occurrence of 1342 within λ, which is impossible. But x lying to the left of s within λ' would cause λ to contain an occurrence of 3124 as witnessed by $jkx(s-1)$. Thus, it must be the case that λ' avoids the patterns in π_8 if λ does, as desired. ☐

Summing (2.12) over $1 \le j \le i-2$ yields the recurrence

$$\begin{aligned} a(n;i) &= a(n-1;i-1) + 2a(n-1;i) + a(n-3)\delta_{i=n-2} \\ &+ \sum_{j=1}^{\min(i,n-2)} \left(a(n-1;j) - a(n-2;j-1) - 2a(n-2;j)\right), \qquad 3 \le i \le n-1. \end{aligned} \tag{2.13}$$

Since $a(n;2) = a(n-1;1) + 2a(n-1;2)$, recurrence (2.13) is seen to hold for $i = 2$ and $n \ge 3$ as well, with $a(n;1) = \#S_{n-1}(231, 213) = 2^{n-2}$ and $a(n;n) = a(n-1)$.

Define the generating functions

$$A(x,y) = \sum_{n\ge1}\sum_{i=1}^{n} a(n;i)x^n y^i$$

and

$$A(x) = \sum_{n\ge1} a(n)x^n.$$

Note that $A(x) = A(x,1)$. The following lemma, valid for arbitrary $a(n;i)$, will be useful. Its proof is routine.

Lemma 2.19.

$$\sum_{n\ge1}\sum_{i=1}^{n}\sum_{j=1}^{i} a(n;j)x^n y^i = \frac{A(x,y) - yA(xy,1)}{1-y}.$$

Using (2.13) for $n \ge 3$ and Lemma 2.19 yields after several algebraic steps the functional equation

$$A(x,y) = xy(1-x)(1-x-xy) - \frac{x\left(x(y+2) + y^2 + y - 3\right)}{1-y}A(x,y)$$

$$+\frac{xy\left(x^2\left(1-y^2\right)+x\left(y^2+3y-1\right)-y\right)}{1-y}A(xy,1). \tag{2.14}$$

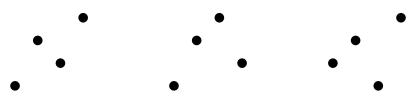
Taking $y=1-x$ in (2.14) implies

$$A(x-x^2,1)=\frac{x(1-x)^2}{1-3x+2x^2-x^3},$$

which gives the generating function $\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}$ for $1+A(x)$. Since $A(xy,1)=A(xy)$, substituting in (2.14) gives the bivariate generating function $A(x,y)$.

2.9. Class 9

$\pi_9=\{1324,1342,2314\}$



Let $d(n;i)$ denote the number of permutations of length n avoiding the three patterns in question and starting with the letter i and let $d(n)=\sum_{i=1}^{n}d(n;i)$. We have the following recurrence formula for the $d(n;i)$.

Lemma 2.20. *If $n\geq 2$, then $d(n;1)=2^{n-2}$ and $d(n;n)=d(n;n-1)=d(n-1)$, with $d(1)=d(1;1)=1$. If $n\geq 4$, then*

$$d(n;i)=2^{n-i-1}d(i-1)+\sum_{\ell=i+1}^{n}\sum_{j=1}^{i-1}d(\ell-1;j), \qquad 2\leq i\leq n-2. \tag{2.15}$$

Proof. That $d(n;n)=d(n;n-1)=d(n-1)$ is clear since neither n nor $n-1$ can start an occurrence of any pattern in π_9. Let $\mathcal{D}_{n,i}$ denote the subset of the permutations of length n enumerated by $d(n;i)$ and let $\mathcal{D}_n=\cup_{i=1}^{n}\mathcal{D}_{n,i}$. That $d(n;1)=2^{n-2}$ follows from the fact that members of $\mathcal{D}_{n,1}$ are synonymous with permutations of length $n-1$ avoiding both 213 and 231 (which are seen to number 2^{n-2}). We now assume $2\leq i\leq n-2$ and show (2.15). We first count members $\alpha\in\mathcal{D}_{n,i}$ in which all elements of $[i+1,n]$ occur to the left of all elements of $[i-1]$, i.e., α that may be decomposed as $\alpha=i\alpha_1\alpha_2$ where α_1 and α_2 are permutations of $[i+1,n]$ and $[i-1]$, respectively. Note that α_1 must avoid both 213 and 231, while α_2 need only avoid the original patterns in π_9. Thus, there are $2^{n-i-1}d(i-1)$ possibilities in this case.

Now assume that the leftmost element j of $[i-1]$ occurs earlier than some element of $[i+1,n]$ within $\alpha\in\mathcal{D}_{n,i}$. Then α must have the form $\alpha=i\alpha_1 j\alpha_2$, where $\alpha_1=n(n-1)\cdots(\ell+1)$ for some $i+1\leq\ell\leq n$. To see this, note first that $i+1$ must occur somewhere to the right of j, for if it occurred to the left of j, then some element x of $[i+1,n]$ occurring to the right of j within α implies that there would be an occurrence of 2314 as witnessed by the subsequence $i(i+1)jx$. Then $i+1$ occurring to the right of j implies any elements of $[i+2,n]$ to the left of j must be in descending order so as to avoid 1342. Finally, if $i<y<n$ lies to the left of j, then so must $y+1$, for otherwise there would be an occurrence of 2314 witnessed by the subsequence $iyj(y+1)$. Thus, α_1 has the stated form. Furthermore, it is seen that the letters in α_2 constitute a member of $\mathcal{D}_{\ell-1,j}$, upon arguing that $j\alpha_2$ avoids the patterns in π_9 if and only if $ij\alpha_2$ does. Conversely, any permutation of the form α above with the stated restrictions on its constituent parts is seen to avoid the patterns in π_9. Considering all possible ℓ and j, it follows that there are $\sum_{\ell=i+1}^{n}\sum_{j=1}^{i-1} d(\ell-1;j)$ members of $\mathcal{D}_{n,i}$ in which some element of $[i+1,n]$ occurs to the right of some element of $[i-1]$. Combining this case with the previous one yields (2.15). □

Let $v(n;y)=\sum_{i=1}^{n} d(n;i)y^i$. Multiplying both sides of (2.15) by y^n, and summing over $2\le i\le n-2$, implies

$$\begin{aligned}v(n;y) &= 2^{n-2}y+(1+y)d(n-1)y^{n-1}+2^{n-1}\sum_{i=2}^{n-2}d(i-1)\left(\frac{y}{2}\right)^i\\ &+\sum_{i=2}^{n-1}y^i\sum_{\ell=i+1}^{n}\sum_{j=1}^{i-1}d(\ell-1;j)-y^{n-1}\sum_{j=1}^{n-2}d(n-1;j)\\ &=2^{n-2}y+(1+y)d(n-1)y^{n-1}+2^{n-1}\sum_{i=2}^{n-2}d(i-1)\left(\frac{y}{2}\right)^i\\ &+\frac{y}{1-y}\sum_{\ell=2}^{n-1}(v(\ell;y)-y^{\ell}v(\ell;1))-y^{n-1}(v(n-1;1)-v(n-2;1)),\ n\ge 3. \qquad (2.16)\end{aligned}$$

Let $v(x,y)=\sum_{n\ge1}v(n;y)x^n$. Then recurrence (2.16) implies

$$\begin{aligned}v(x,y)-v(1;y)x-v(2;y)x^2 &= \frac{2x^3y}{1-2x}(1+v(xy,1))+x(1+y)(v(xy,1)-xy)\\ &+\frac{xy}{(1-x)(1-y)}(v(x,y)-v(xy,1))-x(v(xy,1)-xy)+x^2yv(xy,1),\end{aligned}$$

which may be rewritten as

$$\begin{aligned}&\left(1-\frac{xy}{(1-x)(1-y)}\right)v(x,y)\\ &=\frac{xy(1-x)}{1-2x}+\left(\frac{xy(1-x)}{1-2x}-\frac{xy}{(1-x)(1-y)}\right)v(xy,1). \qquad (2.17)\end{aligned}$$

To solve functional equation (2.17), we use the kernel method and let $y=1-x$ to obtain

$$v(x(1-x),1)=\frac{x(1-x)^2}{1-2x-x(1-x)^2}.$$

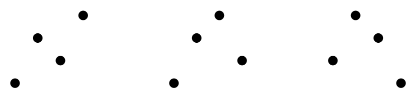
Replacing x with $\frac{1-\sqrt{1-4x}}{2}$ then implies

$$\begin{aligned} 1+v(x,1) &= \frac{\sqrt{1-4x}}{\sqrt{1-4x}-\left(\frac{1-\sqrt{1-4x}}{2}\right)\left(\frac{1+\sqrt{1-4x}}{2}-x\right)} = \frac{2\sqrt{1-4x}}{(2-x)\sqrt{1-4x}-x} \\ &= \frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}, \end{aligned}$$

as desired. (Note that replacing x with $\frac{1+\sqrt{1-4x}}{2}$ leads to a power series whose coefficients are not all positive integers.)

2.10. Class 10

$\pi_{10} = \{1324,\ 1432,\ 2431\}$

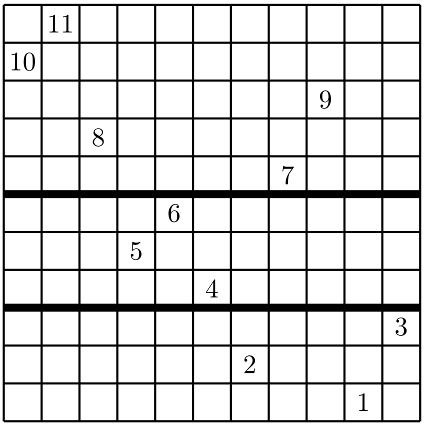


We will count the number $u(n)$ of length-n π_{10}-avoiders directly. The first 3 letters of each pattern in π_{10} form a 132 pattern. So, not surprisingly, 132-avoiders, counted by the Catalan numbers $C(n)$, will figure prominently. Every 132-avoider is a π_{10}-avoider. Let $\mathcal{V}(n)$ denote the set of length-n π_{10}-avoiders that do contain a 132, and set $v(n) = |\mathcal{V}(n)|$. Thus $u(n) = C(n)[\text{ avoids } 132\,] + v(n)[\text{ contains } 132\,]$.

Now suppose acb is a 132 pattern in $p \in \mathcal{V}(n)$. Then every entry of p after b is $< c$ (else a 1324 is present) and $> b$ (else a 1432 or 2431 is present), and the entries after b are increasing (else a 1432 is present). This stringent restriction implies that only one entry, say $b = b(p)$, is the "2" of a 132 in p. Note that if all entries after b in a permutation $p \in \mathcal{V}(n)$ are deleted, the resulting permutation, when standardized, is a *132-ender*, defined to be a π_{10}-avoider that contains a 132 and such that all its 132's end at its last entry.

Our strategy will be to start with a length-k 132-ender p and, viewing it as a permutation matrix, determine how many ways to append $n-k$ increasing entries all lying between the appropriate bounds without introducing a 1324 (we need not worry about introducing a 1432 or 2431 since these new entries are increasing). Then we sum over all k and p.

For a length-k 132-ender p, let b denote its last entry and c the smallest entry that serves as the "3" of a 132. Draw heavy lines above b and below c as in Figure 6. These heavy lines determine the *inner* and *outer* permutations of p, denoted $\text{Inn}(p)$ and $\text{Out}(p)$ respectively: standardize the subpermutation consisting of the entries between the 2 heavy lines to get $\text{Inn}(p)$ and standardize the entries outside the heavy lines to get $\text{Out}(p)$. The original permutation p can be recovered from $\text{Inn}(p)$ and $\text{Out}(p)$ because, as is easily seen, the entries between the heavy lines necessarily form a contiguous block (factor) of p that lies immediately to the left of the leftmost entry $< b$.



Out(p) =
St$(10\,11\,8\,2\,7\,9\,1\,3)$ =
$7\,8\,5\,2\,4\,6\,1\,3$

Inn(p) =
St$(5\,6\,4) = 2\,3\,1$

Figure 6: A 132-ender, $10\,11\,8\,5\,6\,4\,2\,7\,9\,1\,3$, with $b = 3$ and $c = 7$

Any 132-avoider can be an inner permutation, and outer permutations are characterized by the properties (i) is a 132-ender, (ii) the smallest c that serves as the "3" of a 132 is $b+1$ where b is the last entry. Let $\mathcal{A}_m$ denote the set of length-m permutations meeting these two conditions and set $w_0(m) = |\mathcal{A}_m|$.

The number of ways to append $n-k$ increasing entries as specified to a length-k 132-ender p depends only on the 132-avoider q:=Inn(p) and $t := n-k$. Let $w_1(q,t)$ denote this number. Then, refining the count by the length m of Inn(p), we have

$$v(n) = \sum_{k=3}^{n} \sum_{m=0}^{k-3} w_0(k-m) \sum_{q \in S_m(132)} w_1(q, n-k). \tag{2.18}$$

To evaluate the inner sum, we use a bijection from $S_m(132)$ to certain restricted growth sequences. Set $RG_m = \{a_1 a_2 \cdots a_{m+1} : a_1 = 1,\ 2 \le a_i \le a_{i-1}+1 \text{ for } 2 \le i \le m+1\}$. Thus $RG_0 = \{1\}$, $RG_1 = \{12\}$, $RG_2 = \{122, 123\}$, $RG_3 = \{1222, 1223, 1232, 1233, 1234\}$. There is an obvious correspondence between RG_m and primitive Dyck paths of semilength $m+1$ via upstep heights; thus

$$UUDUUDDD \mapsto 1223.$$

The bijection $S_m(132) \to RG_m$ is illustrated in Figure 7 below. Given $q \in S_m(132)$, append $0\ m+1$, and in the matrix diagram, draw a line segment from each nonterminal entry to the next larger entry. Set a_i = number of segments crossing the i-th interior horizontal line. To reverse the map, discard a_1 and set $b_i = a_{i+1} - 1,\ 1 \le i \le m$. Start with 1 and then, for $2 \le i \le m$, build the permutation by successively inserting i in the b_i-th currently available slot (right to left), where

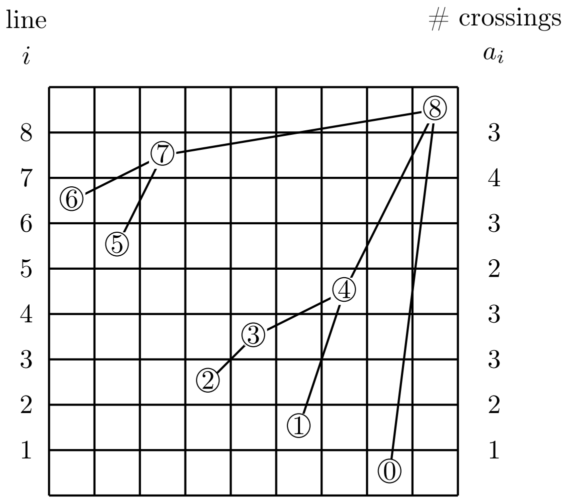


Figure 7: The bijection $S_m(132) \to RG_m$ with $m = 7$: $q = 6572314 \mapsto a = 12332343$

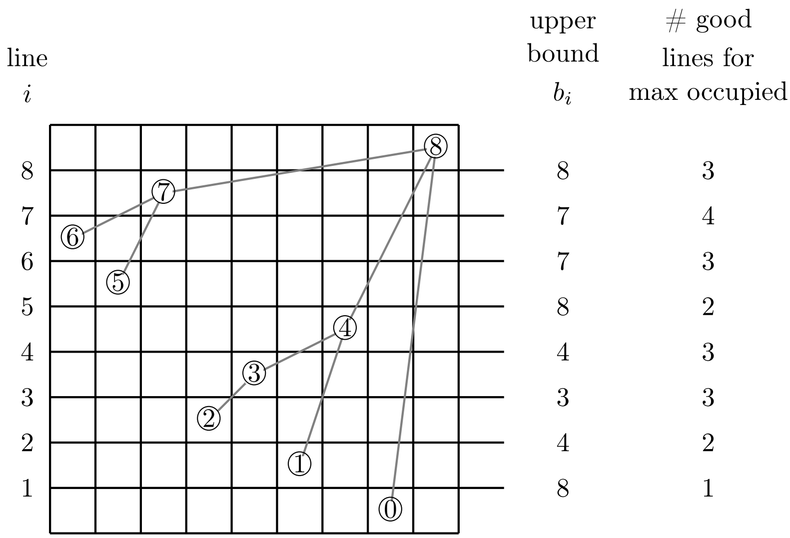


Figure 8: Counting the ways to append entries

available means "won't introduce a 132". Now let us count the number of ways to suitably append t increasing entries to a 132-avoider, using the permutation q of Figure 7 as an example. Since the new entries are increasing, this amounts to inserting t balls into $m+1$ boxes, the boxes being the protruding horizontal lines in Figure 8 above. But there are restrictions. The presence of a ball on line i means all balls lie on or below line b_i, where b_i is the next larger entry after $i-1$ (else a 1324 is present). This implies the upper bounds b_i listed in Figure 8.

Consequently, if i is the largest numbered line containing a ball, then the other $t-1$ balls are constrained to lie on a line j satisfying $j \le i$ and $b_j \ge b_i$. The number of such lines is given in the last column and this column *coincides with* the image $a \in RG_n$ of q under the preceding bijection. So the total number of ways to extend q is $\sum_{i=1}^{m+1} \binom{a_i+(t-1)-1}{t-1}$ using the familiar balls-in-boxes formula.

Hence, with $t := n-k$, the inner sum in (2.18) becomes

$$\begin{aligned}\sum_{q\in S_m(132)} w_1(q,t) &= \sum_{j=1}^{m+1} (\text{total number of } j\text{'s in } RG_m) \times \binom{j+(t-1)-1}{t-1} \\ &= \sum_{j=1}^{m+1} C(m+1-j, 2j-2)\binom{j+t-2}{t-1},\end{aligned} \tag{2.19}$$

where $C(n,k) = \frac{k+1}{2n+k+1}\binom{2n+k+1}{n} = \binom{2n+k}{n} - \binom{2n+k}{n-1}$ is the generalized Catalan number that counts nonnegative lattice paths of $n+k$ upsteps and n downsteps. The second equality in (2.19) is left as an exercise for the reader.

Next, we compute $w_0(m) = |\mathcal{A}_m|$. A 132-ender with consecutive bc arises by suitably appending an entry to a 132-avoider. As Figure 9 illustrates, you can append an entry on any non-top line except just below a LR min. There are

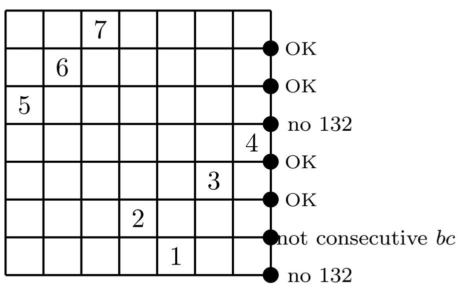


Figure 9: Constructing 132 enders with consecutive bc

$N(m-1,k)$ (Narayana number, $N(n,k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$) 132-avoiders of length $m-1$ with k LR minima, each of which contributes $m-1-k$ elements to $\mathcal{A}_m$. Hence $w_0(m) = \sum_{k=1}^{m-1}(m-1-k)N(m-1,k) = \binom{2m-3}{m-3}$.

So (2.18) becomes

$$\begin{aligned} v(n) &= \sum_{k=3}^{n}\sum_{m=0}^{k-3}\binom{2(k-m)-3}{k-m-3}\sum_{j=1}^{m+1}C(m+1-j,2j-2)\binom{j+n-k-2}{n-k-1} \\ &= \sum_{k=3}^{n}\sum_{j=1}^{k-2}\binom{j+n-k-2}{n-k-1}\sum_{m=j-1}^{k-3}\binom{2(k-m)-3}{k-m-3}C(m+1-j,2j-2) \\ &= \sum_{k=3}^{n}\sum_{j=1}^{k-2}\binom{j+n-k-2}{n-k-1}\binom{2k-2}{k-j-2}. \end{aligned} \tag{2.20}$$

The last equality follows from the identity

$$\sum_{i=1}^{n-k}\binom{2i+1}{i-1}C(n-k-i,2k)=\binom{2n+2}{n-k-1},$$

which has a simple combinatorial proof: it counts lattice paths of $n+k+3$ upsteps and $n-k-1$ downsteps, starting at the origin, by the x-coordinate, $2i+1$, of the last vertex at height 3. This vertex is the left endpoint of an upstep whose removal splits the path into a pair of paths counted by the summand on the left.

Now let us find the generating function for the sequence $u(n)$, that is, $U(x)=\sum_{n\geq 0}u(n)x^n$. By the above, we have

$$\begin{aligned} U(x) &= \sum_{n\geq 3}v(n)x^n+\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n \\ &= \sum_{n\geq 4}\left(\sum_{k=3}^{n-1}\sum_{j=1}^{k-2}\binom{j+n-k-2}{n-k-1}\binom{2k-2}{k-2-j}x^n\right) \\ &\quad +\sum_{n\geq 3}\binom{2n-2}{n-3}x^n+\frac{2}{1+\sqrt{1-4x}} \\ &= \sum_{k\geq 3}\sum_{j=1}^{k-2}\binom{2k-2}{k-2-j}\left(\sum_{n\geq k+1}\binom{j+n-k-2}{n-k-1}x^n\right) \\ &\quad +\frac{16x^3}{\sqrt{1-4x}(1+\sqrt{1-4x})^4}+\frac{2}{1+\sqrt{1-4x}} \\ &= \sum_{j\geq 1}\sum_{k\geq j+2}\binom{2k-2}{k-2-j}\frac{x^{k+1}}{(1-x)^j} \\ &\quad +\frac{16x^3}{\sqrt{1-4x}(1+\sqrt{1-4x})^4}+\frac{2}{1+\sqrt{1-4x}} \\ &= \sum_{j\geq 1}\frac{4^{j+1}x^{j+3}}{(1-x)^j\sqrt{1-4x}(1+\sqrt{1-4x})^{2j+2}} \end{aligned}$$

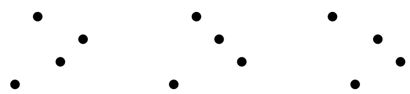
$$+\frac{16x^3}{\sqrt{1-4x}(1+\sqrt{1-4x})^4}+\frac{2}{1+\sqrt{1-4x}}$$
$$=\frac{16x^4(1-2x+\sqrt{1-4x})}{\sqrt{1-4x}(1+\sqrt{1-4x})^4((1-x)\sqrt{1-4x}+1-5x+2x^2)}$$
$$+\frac{16x^3}{\sqrt{1-4x}(1+\sqrt{1-4x})^4}+\frac{2}{1+\sqrt{1-4x}},$$

which implies

$$\sum_{n\geq 0} U(n)x^n = \frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}.$$

2.11. Class 11

We use the representative triple $\pi_{11} = \{1423, 1432, 4132\}$



Let $A_n = S_n(\pi_{11})$. Let $\sigma \in A_n$ with $n \geq 1$. Then σ can be decomposed as either which can be described as follows.

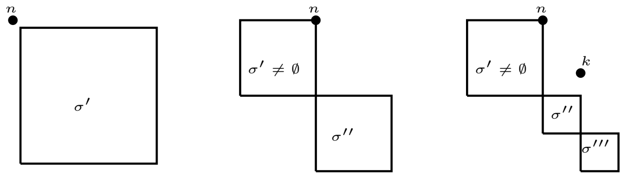


Figure 10: Decompositions

Lemma 2.21. *Let $n \geq 2$. A permutation π of $[n]$ avoids π_{11} if and only if either*

- $\pi = n\pi'$ *such that π' is a permutation of $[n-1]$ that avoids* 132*; or*
- $\pi = \pi' n \pi''$ *such that $\pi' > \pi''$, where π' is a non-empty permutation of $[n-j+1, n-1]$ that avoids π_{11} and π'' is a permutation of $[n-j]$ that avoids* 132*; or*
- $\pi = \pi' n \pi'' k \pi'''$ *such that $\pi' > \pi'' > \pi'''$, where $\pi' k$ is a permutation of $[n-j+1, n-1]$ avoiding π_{11} of length at least two such that $k \neq n-j+1$, π'' is a permutation of $[d+1, n-j]$ that avoids* 132*, and π''' is a permutation of $[d]$ that avoids* 132.

Let $A(x) = \sum_{n\geq 0} \#A_n x^n$. Using Lemma 2.21, we obtain

$$A(x) = 1 + xC(x) + x(A(x)-1)C(x) + x(A(x)-1-xA(x))C(x),$$

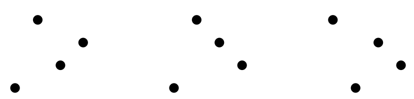
where $A(x)-1-xA(x)$ is the generating function for the number of permutations $\sigma=\sigma_1\cdots\sigma_n$ of A_n, $n\geq 2$, such that $\sigma_n\neq 1$. Thus, we can state the following result.

Theorem 2.22. *The generating function for the number of permutations of length n that avoid π_{11} is given by*

$$\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}.$$

2.12. Class 12

We use the representative triple $\pi_{12}=\{2314, 2341, 3124\}$



Let $c(n;i,j)$ denote the number of permutations of length n avoiding the patterns in π_{12} in which the first letter is i and the second is j. For $n\geq 2$, define $c(n;i)=\sum_{j=1,j\neq i}^{n} c(n;i,j)$ and $c(n)=\sum_{i=1}^{n} c(n;i)$, with $c(1)=c(1;1)=1$. The values of the array $c(n;i,j)$ for $n\leq 3$ clearly are the same as those given above for $a(n;i,j)$.

If $n\geq 4$, then the array $c(n;i,j)$ satisfies the following relations.

Lemma 2.23. *If $1\leq i\leq n-1$, then $c(n;i,n)=c(n;n,i)=c(n-1;i)$, with $c(n;1,i)=c(n;i,i-1)=c(n-1;i-1)$ for $1<i\leq n$. If $2\leq i<j<n$, then $c(n;i,j)=0$. If $1\leq j<i-1<n-1$, then*

$$c(n;i,j)=c(n-1;i,j)+c(n-1;i-1,j-1)+\sum_{k=1}^{j-2} c(n-1;j,k). \tag{2.21}$$

Proof. Let $\mathcal{C}_n$ denote the subset of the permutations of length n avoiding the patterns in π_{12} and $\mathcal{C}_{n,i,j}$ the subset of $\mathcal{C}_n$ enumerated by $c(n;i,j)$. The first statement is clear since a letter n in either the first or second position is seen to be extraneous concerning the avoidance of the patterns in π_{12}, as is the letter 1 within members of $\mathcal{C}_{n,1,i}$ and the letter $i-1$ within members of $\mathcal{C}_{n,i,i-1}$. Permutations of length at least four starting with the letters i, j where $2\leq i<j<n$ must contain an occurrence of either 2314 or 2341, whence $c(n;i,j)=0$ in these cases. We now show (2.21). To do so, consider the third letter k within a member of $\mathcal{C}_{n,i,j}$ where $1\leq j<i-1<n-1$. The letter k cannot belong to $[i+1,n-1]$, for if it did, then there would be an occurrence of 2314 or 2341, and it cannot belong to $[j+1,i-1]$, for if it did, then 3124 would occur. Thus, we must have $k=n$ or $k\in[j-1]$, and the first two terms on the right-hand side of (2.21) are seen to account for the cases in which $k=n$ or $k=j-1$, respectively.

So let us assume $k \leq j-2$. Given $\lambda \in \mathcal{C}_{n-1,j,k}$, let λ' be the permutation obtained from λ by writing the letter i before λ and increasing all elements of $[i, n-1]$ within λ by one. We will show that λ avoiding the patterns in π_{12} implies λ' does. Suppose, to the contrary, that λ' contains an occurrence of some pattern $\rho \in \pi_{12}$ and that ρ is realized within λ' by the subsequence $i\ell rs$. First assume $\rho = 3124$. Note that one may take $\ell \leq k$ within an occurrence of ρ in this case, for if $\ell > k$, one may replace ℓ with k. Furthermore, observe that we must have $r > j$, for if not, then λ would contain ρ with the subsequence $j\ell r(s-1)$, which is impossible. Now consider the position of any $y \in [k+1, j-1]$. If y lies (i) to the right of s, (ii) between r and s, or (iii) to the left of r, then there would be an occurrence within λ of 2341, 2314, or 3124, respectively, as witnessed by the subsequences $jr(s-1)y$, $jry(s-1)$, or $jkyr$, with each scenario being impossible. This implies $\rho = 3124$ is not possible.

Now assume $\rho = 2314$. Note that $r > j$, for otherwise λ would contain 2314 with $j(\ell-1)r(s-1)$. But then $r > j$ implies λ' contains an occurrence of 3124 with $ijrs$, which is impossible by the preceding case. Finally, assume $\rho = 2341$. If $y \in [k+1, j-1]$, then λ would contain an occurrence of 2341, 2314, or 3124, respectively, as witnessed by the subsequences $j(\ell-1)(r-1)y$, $j(\ell-1)y(r-1)$, or $jky(\ell-1)$, depending on whether y lies (i) to the right of r, (ii) between ℓ and r, or (iii) to the left of ℓ. Thus, $\rho = 2341$ is also not possible, which implies λ' avoids the patterns in π_{12} if λ does, as desired. $\square$

Note that (2.21) implies for $2 \leq i \leq n-1$,

$$c(n;i) = c(n-1;i-1) + c(n-1;i) + \sum_{j=3}^{i}(c(n-1;j) - c(n-2;j-1) - c(n-2;j)), \tag{2.22}$$

with $c(n;n) = c(n;1) = c(n-1)$.

Define $C_n(v) = \sum_{i=1}^{n} c(n;i)v^{i-1}$. Multiplying both sides of (2.22) by v^{i-1}, and summing over $2 \leq i \leq n-1$, yields

$$\begin{aligned} C_n(v) &= (1+v^{n-1})C_{n-1}(1) + (1+v)C_{n-1}(v) - (1+v^{n-1})C_{n-2}(1) \\ &\quad + \frac{1}{1-v}(C_{n-1}(v) - C_{n-2}(1) - v^{n-1}C_{n-1}(1) + v^{n-1}C_{n-2}(1)) \\ &\quad - \frac{1+v}{1-v}(C_{n-2}(v) - C_{n-3}(1) - v^{n-2}C_{n-2}(1) + v^{n-2}C_{n-3}(1)) \\ &\quad - \frac{v - v^{n-1}}{1-v}C_{n-3}(1) - v^{n-2}(C_{n-2}(1) - C_{n-3}(1)), \qquad n \geq 3, \end{aligned}$$

with $C_0(v) = C_1(v) = 1$ and $C_2(v) = 1 + v$.

Define $C(x,v) = \sum_{n\geq 0} C_n(v)x^n$. Multiplying both sides of the last recurrence by x^n, and summing over $n \geq 3$, we obtain

$$\frac{(1-x-xv)(1-x-v)}{1-v}C(x,v) = (1-x)^2 - vx + \frac{x(1-x)(1-x-v)}{1-v}C(x,1)$$

$$- \frac{vx(1 - vx - 2x + vx^2)}{1 - v} C(xv, 1).$$

Substituting $v = 1 - x$ in the preceding functional equation yields $C(x(1-x), 1) = \frac{1-2x}{(1-x)^3 - x^2}$, which implies

$$C(x, 1) = \frac{2(1 - 4x)}{2 - 9x + 4x^2 - x\sqrt{1 - 4x}}.$$

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