

# Introduction of differential calculus in the class 10 with graphical calculator

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## Abstract

In the German mathematical educational system the graphical calculator is not only allowed but also recommended in secondary grammar school. Therefore in the Deutsche Schule Budapest, where the graphical calculator based on German curriculum is also allowed and required, an experiment were conducted with two groups in grade 10. This experiment was performed in these groups for 46 lectures. The parallel teaching with and without graphical calculator was the main emphasis in the lectures. We consistently used the “old” with the “new” method simultaneously. The aim of the experiment was to determine how the graphical calculator can be helpful in the mathematics courses when it is used for introduction of differential calculus. To apply calculators for mathematical problems secondary-school students have to understand the meaning and the concepts of mathematical formulas confidently, they have to convert and interpret the parameters and substitute them into the right formulas. Besides, they have to know correctly the functions of the calculator in practice. Even better students can have difficulty in using the calculator because they need different notations on paper and booklet operation and on the software or graphical calculator. While it has been suggested that appropriate use of the graphical calculator can support students in differential calculus, the following issues were considered: how and how deep the differential calculus should be instructed with and without graphical calculator so that the students can build the right concept image.

*Keywords:* difference quotient, differential quotient, graphing calculator, representations, secondary grammar school

*MSC:* 97D43, 97D53, 97I43, 97R23

## 1. Introduction

Up until now the graphical calculator has been introduced in a lot of countries, including Germany, but unfortunately not in Hungary. Many studies show (see [2, 5, 11]) that the graphical calculator can have considerably important role in learning mathematics and can improve the pupils' knowledge and skills e.g. in the following areas: concept development, problem solving and computation skills. "*Using graphing calculators in mathematics education brings also new methods of work – especially the possibility of exploration and modelling of mathematical problems, multiple representation of mathematical problems (numerical, algebraic, graphic, algorithmic representation) and graphic support of the results obtained by algebraic procedures.*" (See [13].)

Earlier, before using GC in the lessons the student had no opportunity to see the dynamic of e.g. difference quotient. The teacher drew a function on the blackboard with more secant lines. The students had to be able to imagine the moving these in tangent line and to understand the mathematical meaning of their concept in order to apply it by calculating. With GC it is no problem to realise the connection between e.g. difference quotient and derivative. However with this technology pupils have not to calculate so much, as well. The pupils calculating with GC can use modern IT technique precisely but it often realised they have no idea what kind of mathematics exist in the background. The conducted experiment was *innovative* because we consistently used the "old" with the "new" method simultaneously. That means, pupils had to know the mathematical meaning of the new concepts, to work and to calculate with them without GC. At the same time they had to use GC and to know their necessary functions in this topic. During the lessons and in the post-tests pupils solved the tasks with and without GC.

This experiment evaluated two mathematical domains: the concept of difference and differential quotient and the increase and decrease of function in a given interval, concerning the fact that representation has very important role. The appropriate concept image of the difference quotient can be developed with dual representation (see [16]). Pupils have to learn changing between the enactive, iconic, and later the symbolic meaning of representation, and applying them to problem solving. In this process the graphical calculator can be helpful, according to Kamarulhaili and Sim (see [8]): "*The cognitive gain in number sense, conceptual development, and visualization can empower and motivate students to engage the true mathematical problem solving at a level previously denied to all but the most talented. The calculator is an essential tool to all students in mathematics.*"

Nowadays, practical strategies in the mathematical teaching system are widely encouraged, and sometimes, overemphasised, thus during the experiment the following important questions were considered: Is it possible that these technical applications overshadow the theoretical knowledge? Is it possible that students cannot apply mathematical relations to different contexts, if the theory was not evidently recognizable and can forget the basic mathematical operations using the opportunities provided by the graphical calculator? How could they be trained to

solve basic exercises properly also without a calculator? Considering all angles of the questions, the answers to the first two questions pointed in the same direction. It is often observed that if pupils do not get sufficient theoretical background, most of them do not recognise the basic relations in practical exercises, and they cannot generalise it. If the same question that they have already known were asked in different context they cannot recognise the relation between them, and generally, they cannot use the rules when the question is asked in different words. Meir Ben-Hur basically defined the five phases of learning Practice, De-contextualisation, Encapsulating a generalization in words, Re-contextualisation, Realisation (see [3]). The third phase, encapsulating process, where generalization has to be reinforced both in words and mathematically at the same time was highlighted during the experiment.

## 2. Theoretical background

### 2.1. The role of representation

According to the developmental theory by Bruner, the formal operational stage begins at about age 11. As adolescents enter this stage, they gain the ability to think in an abstract manner, the ability to combine and classify items in a more sophisticated way, and the capacity for higher-order reasoning (see [4]).



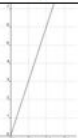
| Modes:          | Enactive   | Iconic  | Symbolic   |
|-----------------|--|---|--|
| Natural numbers | One pair of cherry   |   | 2, two, II   |
| Geometry        | Kite   |  | Deltoid  |
| Functions       | A walker goes 3 km an hour. What is the contact between time and trip? |  | $f: \mathbf{R}^{+0} \rightarrow \mathbf{R}, f(x) = 3x$ |

Figure 1: Modes of representation

Based on this idea, it should be expectable that tenth grade (16 years old) pupils understand and apply discretely phrased definitions and theorems. Generally many of the most talented students can reach this stage after a while, however even they need a lot of support during the process. This can be a well-chosen introductory exercise and an adequate representation. It is obvious in the case of functions where these three models of representation, enactive representation (action-based), iconic representation (image-based), and symbolic representation (language-based) can be efficiently connected (see [12]). Although the sequence of representations is

very important, it is only worth going on the next level when the previous one is already well-understood.

## 2.2. Using information technology in teaching

The technological advancement makes the usage of the tools of information technology and the media possible. Today the range of these tools is rather diverse. We may talk about different computer applications, software, educational videos, interactive boards. In recent times the easily portable tools have come to the foreground. These might be “simple” or graphic calculators, voting-machines, and we can download several useful mathematical and scientific applications even on smart phones.

Our graphical calculator, TI-Nspire CAS, provides several functions. The main functions are:

- Calculator: Perform computations and enter expressions, equations and formulas in proper math notation.
- Graphs: Plot and explore functions, equations and inequalities, animate points on objects and graphs, use sliders to explain their behaviour and more.
- Geometry: Construct and explore geometric figures and create animations.
- Lists & Spreadsheet: Perform mathematical operations on data and visualise the connections between the data and their plots.
- Notes: Enter notes, steps, instructions and other comments on the screen alongside the math.
- Data & Statistics: Summarise and analyse data using different graphical methods such as histograms, box plots, bar and pie charts and more.
- Vernier Data Quest Application: Create a hypothesis graphically and replay data collection experiments all in a single application.

The TI-Nspire CAS handheld’s innovative capabilities support teaching strategies that research has found accelerate understanding of complex mathematic and scientific concepts. Multiple representations of expressions in problems are presented simultaneously, enabling students to visualise how algebraic, graphical, geometric, numeric and written forms of those expressions relate to one another.<sup>1</sup> During the lessons we used the advantages of simultaneously presentations entirely.

Tablets or smart phones can have the similar function like TI-Nspire CAS depending on the used applications. There are many applications, some of them are gratis -like GeoGebra, some of them not -like e.g. WolframAlpha. This application gives functions in mathematics tool like: elementary mathematics, numbers,

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<sup>1</sup><https://education.ti.com/en/us/products/calculators/graphing-calculators/ti-nspire-cas-with-touchpad/tabs/overview>

plotting, algebra, matrices, calculus, geometry, trigonometry, discrete mathematics, number theory, applied mathematics, logic functions, definitions. This software is very useful and user friendly. However, similar experiment can be run on computers or other mobile devices theoretically, but using mobile devices during the lessons have disadvantages. Students have opportunities to use other applications – e.g. games and to make on-line communication which is not allowed not even during the tests.

Timo Leuders claims that applying these new technologies may be especially important in mathematics: “*Neue Technologien und neue Medien (gemeint ist meist: Computer) bieten für den Mathematikunterricht – mehr noch als die meisten anderen Schulfächer – die Chance zu einer grundlegenden inhaltlichen und methodischen Reform. Sie ermöglichen eine Entlastung von Routinearbeiten und bahnen daher exploratives und kreatives Arbeiten, ebenso die Behandlung realistischer Anwendungssituationen und das Vernetzen von Inhalten.*”<sup>2</sup> (See [10].) The question is how much, or to what extent is it necessary and possible to use these new opportunities in teaching. According to Tulodziecky, the different tool of information technology have to be used as support and encouragement in school education, if the teaching process is problem, decision and organisation oriented (see [19]). Tulodziecky considers the application of the media especially important in five cases:

1. Difficult exercises with decomplex initial conditions;
2. If we want to exemplify the goal or the route to the solution;
3. If the individual or cooperative work form comes to the foreground while solving a difficult task;
4. When comparing different modes of solution;
5. During the application of the theory learned and reflections to them.

In our experience, pupils use the graphic calculator with pleasure, and they even use such applications that they do not need. Finally, we would like to mention the viewpoints of Erwin Abfalterer, which have to be considered by all means when planning a lesson with the computer (graphic calculator), so that the lesson flows with the greatest efficiency in the time available:

1. The software has to be prepared and tested;
2. The flow of the lesson is planned and the goal are set;
3. The exercises are given, the role of the teacher is clear;
4. A short feedback always needs to be possible (see [1]).

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<sup>2</sup>“*New technologies and new media (is usually meant: computer) provide for teaching mathematics - more so than most other school subjects – the opportunity for a fundamental substantive and methodological reform. They provide relief from routine tasks and thus pave explorative and creative work, as well as the treatment of realistic situations and use the cross-linking of content.*”

### 2.3. About the use of graphical calculator

According to Horton graphical calculators can be helpful to make connection among representations in the mathematics education and therefore it can “*permit realism through the use of authentic data*” (see [6]). In 2000 the National Council of Teachers of Mathematics summarised the results from many resources (see [14, 15, 18]). Data have shown that using graphical calculators had potential benefits:

- *Speed*: after the students could handle the tool appropriately, they had opportunity to compute, graph, or create a table of values quickly.
- *Leaping Hurdles*: without technology, it was nearly impossible for students who had few skills and little understanding of fractions and integers to study algebra in a meaningful way. Consequently, lower level high school courses often became arithmetic remediation courses. With technology, all students have opportunity to study rich mathematics. They can use their calculators to perform the skills that they are unable to do themselves.
- *Connections*: sophisticated use of graphing calculator promotes students to make connections among different representations of mathematical models. Users can quickly manoeuvre among tabular, graphical, and algebraic forms.
- *Realism*: No longer are teachers restricted to using contrived data that lead to only integer or other simplistic solutions. Graphical calculators permit the creation of several types of best-fitting regression models. This capability allows data analysis to become integrated within the traditional curriculum; the tedium and difficulty of calculating a best-fit model are no longer factors in introducing data analysis into the curriculum. (pp. 24-32)

Tiwari also proved that the connection between algebraic and geometric representations with graphical calculators can be deeper in calculus; the graphical calculator can support the understanding “when it is used as a supplementary instructional tool in achieving conceptual understanding and enhancing problem solving abilities of students in learning differential calculus” (see [17]). Van Streun, Harskamp and Suhre showed (see [21]) that the use of graphical calculators could lead to changes in students’ approaches in problem solving. These positive changes have affected students’ successfulness. Jones got similar results (see [7]). He thought that using the graphical calculator pupils can approach problems graphically, numerically and algebraically. Ng Wee Leng found that the use of various problem-solving approaches can support students’ visualization in order to find the solution and allow them to explore problem situations that they probably could not handle otherwise (see [9]).

## 3. Research methodology

In the Deutsche Schule Budapest, the curriculum of the German Baden- Württemberg province is applied. There, smart boards can be found in every classroom,

therefore the students have the opportunity to use the software GeoGebra during the lessons. They also have to download it at home, and to do homework with the help of the software in lower classes as well. With that method, they can practice the sketching functions. Later on, in class 10, it can support them using the graphical calculator.

TI-nspire CAS type graphical calculators were used by the pupils and they can use it constantly in class and at home as well. The tenth grade mathematics curriculum starts with the topic functions. As a first step the most important definitions were reviewed followed by five more lessons soon afterwards. On these lessons the notions of the difference quotient, the derivative, derivative function, and the derivative rules were introduced together with some simple related definitions. In this chapter the graphical calculator and handmade calculation were used in parallel.

During the lessons we always had the opportunity to communicate with pupils and to help them. Furthermore, if some of them had problems, they could help each other as well. We had altogether 46 lessons for the examined topic – 11 weeks, four lessons a week. During this period the following chapters were covered:

- Lessons 1–4: Reviewing functional elements. Domain, range, zero point, function value. Practice.
- Lesson 5–8: Introducing the difference quotient. Practice.
- Lessons 9–12: The differential quotient. Practice.
- Lessons 13–16: Calculation of the differential quotient. Practice.
- Lessons 17–20: Derivative function. Practice.
- Lessons 21–24: Derivative rules. Practice.
- Lessons 25–26: Summary, practice.
- Lessons 27–28: First test.
- Lessons 29–32: Intervals of increase and decrease. Practice.
- Lessons 33–36: Maxima and minima of function (local, absolute). Practice.
- Lessons 37–38: Sketch polynomial functions. Practice.
- Lessons 39–42: All day problems with (polynomial) functions. Practice.
- Lessons 43–44: Summary, practice. Practice.
- Lessons 45–46: Second test.

## 4. Discussion of Experiment

The 10a class consisted of 18, the 10b consisted of 19 pupils in the observation period. The mathematical knowledge of the pupils was considered to be common. The averages of the mathematics grades was 2.73 (standard deviation: 1.46) and 2.89 (1.24) in the two classes (according to the German system, where 1 is the best and 6 is the worst.) The first test was written by 12 and 13 pupils, respectively, the rest of the class was sick on the day of the test. The averages were 2.50 (standard deviation: 1.26) and 2.57 (1.17). After the 12 students became healthy again, they wrote a similar test. The results of this post-test (2.71, standard deviation: 1.22) are not included in the average of the experiment. The second test was written by every pupil. The averages of the second test were 2.37 (standard deviation: 1.21) and 2,58 (standard deviation: 1.62). The grading scheme was following:

|         |      |    |      |      |      |      |      |    |      |
|---------|------|----|------|------|------|------|------|----|------|
| To ...% | 95   | 90 | 85   | 80   | 75   | 70   | 65   | 60 | 55   |
| Grade   | 1+   | 1  | 1-   | 2+   | 2    | 2-   | 3+   | 3  | 3-   |
| Value   | 0.66 | 1  | 1.33 | 1.66 | 2    | 2.33 | 2.66 | 3  | 3.33 |
| To ...% | 50   | 45 | 40   | 33.3 | 26.7 | 20   | 0    |    |      |
| Grade   | 4+   | 4  | 4-   | 5+   | 5    | 5-   | 6    |    |      |
| Value   | 3.66 | 4  | 4.33 | 4.66 | 5    | 5.33 | 6    |    |      |

Table 1: Grading scheme

After the first 26 lessons the students wrote the first 90-minute test. It was not allowed to use a graphical calculator in part A. In this part, the tasks could be solved with simple calculation, however, the students had to know the exact definition. Since part B could not be solved without graphical calculator, the pupils had to handle it confidently.

### 4.1. First test

#### *Group A*

##### *Part A*

***It is NOT ALLOWED to use a GC!***

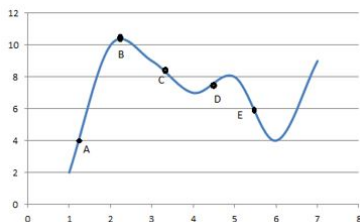
1. Given the function  $f: f(x) = \sqrt{x}$  ( $\in R^+$ ). Give the derivative function and prove it. (1+3)
2. Given the function  $f: f(x) = \frac{1}{2}x^2 + 1$ .
  - a) Give the difference quotient in interval  $[0; 2]$ . (3)
  - b) Give the geometric meaning of the result. (1)
  - c) Give the differential of  $f$  at  $x_0 = 2$  with  $h \rightarrow 0$ . (3)
  - d) Give the derivative of function. (2)



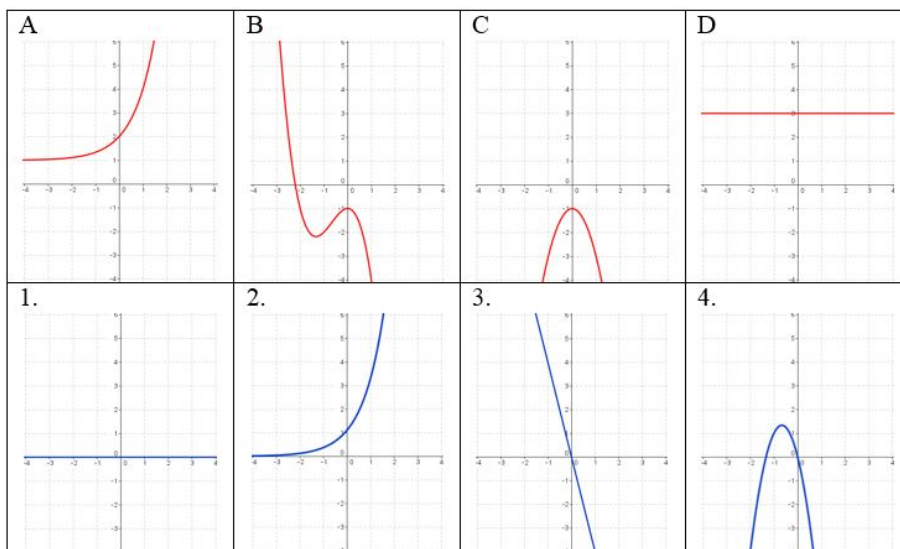
e) Give the geometric meaning of the result in c). (1)

3. See the function graph in interval ( $D = [1; 7]$ ) and answer the questions.

- Give the intervals when the slope of the function (of the tangent line) is positive. (3)
- Arrange the slopes of the tangents at the points marked in a growing order. (4)
- Draw the approximate tangent lines at the points B, C and D, and evaluate their slopes. (1.5+1.5)



4. Couple the graphs of the functions (red) with the derivative functions (blue) and explain shortly why only the specified pairs were chosen. (4+4)



### Group A

#### Part B

*It is ALLOWED to use a GC!*

5. Give the derivative functions of the following functions, and calculate the slope at the point  $x_0 = 3$ . Suggestion: calculate the derivative function and the value function without a GC, and then you can check them with a GC.

a)  $f(x) = 3x^2 - 2x + 3$  (3+2)

b)  $g(x) = \frac{2}{x}x^{-3}$  (4+2)

6. Determine the tangent term of the function using GC:

$$f: f(x) = 2x^4 - 3x^2 + x - 5$$

at  $x_0 = 2$ . Describe briefly how you proceeded! Sketch the function and the tangent line using the GC on the sheet. (3+2+3)

7. Given the function  $f: f(x) = -2x^3 + 5x + 4$ .

a) Plot the function AND its derivative function with a GC, and sketch them appropriately on the sheet. (3+3)

b) Explain briefly why the derivative function is just seen at the zero points, or why the derivative function is only positive / negative where you see it. (3)

Just slight differences were measured between the two groups (group A and group B) and between the two classes, therefore we assessed the results of the two classes altogether – about the detailed results see Várady [20] and appendix. Pupils wrote the test with good outcomes which was surprisingly unexpected: the result in the theoretical part (Part A) was 71.1%. In this part lots of questions were about the meaning of tangent line and about the slope of functions without using the graphical calculator. In the graphical calculator part (Part B) the result was 78%, where pupils got questions inter alia about the meaning of tangent line and about the slope of functions. Both of them were considered a good rate in an average class and they were also balanced.

|      |      |      |      |      |      |      |      |      |      |                |      |
|------|------|------|------|------|------|------|------|------|------|----------------|------|
| 1.   |      | 2.   |      |      |      |      | 3.   |      |      | 4.             |      |
| Satz | Bew. | a)   | b)   | c)   | d)   | e)   | a)   | b)   | c)   | a)             | b)   |
| 84.0 | 46.7 | 69.3 | 68.0 | 53.3 | 72.0 | 68.0 | 73.3 | 69.0 | 71.3 | 95.0           | 78.0 |
| 5.   |      |      |      | 6.   |      |      | 7.   |      |      | <b>Average</b> |      |
| a1)  | a2)  | b1)  | b2)  | a)   | b)   | c)   | a1)  | a2)  | b)   |                |      |
| 82.7 | 80.0 | 68.0 | 60.0 | 74.7 | 76.0 | 85.3 | 92.0 | 84.0 | 74.7 | <b>75.2 %</b>  |      |

Table 2: Results of first test

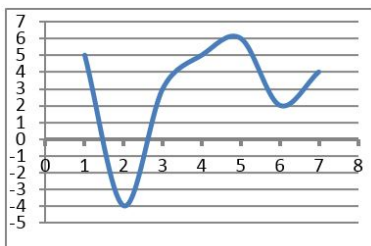
## 4.2. Second test

The second chapter included the most important characteristics of functions and their ways of calculation - domain, range, interval of increase, interval of decrease, constant, maximum, minimum, zero of a function, end behavior. During the lessons and in the homework pupils were encouraged to do exercises with and also without the graphical calculator. There were some situations where they were not allowed to use it, and there were some, where the function describing the process was such that they were not able to do the task without a calculator. Of course, pupils wrote a test at the end of the chapter, where they were not allowed to use the calculator for some exercises, one task had to be done with and also without the calculator, and the last exercise could only be done with the help of the calculator. There were two groups in which the tasks were similar to each other. One of the task sheets looked like the following:

### Part A

**It is NOT ALLOWED to use a GC!**

1. Give the intervals of increase and decrease and the maxima and minima of this function. The function is defined in the interval  $[1; 7]$ .



2. Given is the function:  $f(x) = -x^3 + 6x^2$ .
  - a) Give the maximal domain!
  - b) Calculate the zero points!
  - c) Calculate the intervals of increase and decrease and the maxima and minima of this function with the derivative function.
  - d) Determine the  $y$  intercepts of the curve.
  - e) Sketch the function!

### Part B

**It is ALLOWED to use a GC!**

3. Determine the zero points of function! First you have to calculate without the calculator, after that you have to check them with GC. Describe your steps with GC!

$$f(x) = (x^3 - 20)(x^4 - 28x^2 + 75)$$

4. The next function gives the profile of a landscape in interval  $[1; 7]$ . The value  $x$  is km, the value of  $f(x)$  is 10 meters.

$$h(x) = 0.3x^4 - 5.1x_3 + 29.5x^2 - 65.1x + 45.3$$

- Sketch the function with GC and draw the curve on the sheet.
- Calculate the maxima and minima of the landscape and their values with GC.
- Give the intervals of increase and decrease.

### 1st task:

Perfectly done by 8 pupils. Nine pupils did not consider the closed interval, so two end points were missing from the maxima, but the other solutions were good. The remaining 3 pupils had other mistakes as well, they defined the intervals with wrong starting and end points and/or the maxima and minima were not good either.

|   |   |
|---|---|
| <p>① absolutes Maximum <math>(x 5)</math>, <math>(7 4)</math><br/>         lokales Minimum <math>(2 4)</math><br/>         Sattelpunkt <math>(4 5)</math><br/>         lokales Maximum <math>(5 4)</math><br/>         lokales Minimum <math>(6 2)</math><br/> <math>x \in ]5; 4[</math> streng monoton fallend<br/> <math>x \in ]4; 6[</math> streng monoton steigend<br/> <math>x \in ]6; 2[</math> streng monoton fallend<br/> <math>x \in ]2; 4[</math> streng monoton steigend</p> | <p>① <math>x \in ]2; 4[</math><br/> <math>x \in ]0; 2[</math>; <math>2; 3[</math> streng monoton steigend<br/> <math>x \in ]3; 4[</math> streng monoton steigend<br/>         Hochpunkt (absolut)<br/> <math>x \in ]4; 5[</math> streng monoton fallend<br/> <math>5 =</math> Tiefpunkt (absolut) (wie groß?)<br/> <math>x \in ]5; 6[</math>; <math>6; 7[</math> streng monoton fallend<br/> <math>6 =</math> Hochpunkt (wie groß?)</p> |
|---|---|

Figure 2: Pupils' solutions

### 2nd task:

- Everybody answered well.
- 18 pupils gave the derivative function well, 15 among them counted the zero points of derivative function well. The 3 pupils missed 0 as zero point (they divided by  $x$  without checking the opportunity  $x = 0$ ). Two pupils derived in the wrong way.
- Thirteen among fifteen pupils drew the table well, two of them wrongly substituted back into the  $f(x)$  function, so they got partly wrong results. 13 pupils did this part well (Figure 3).

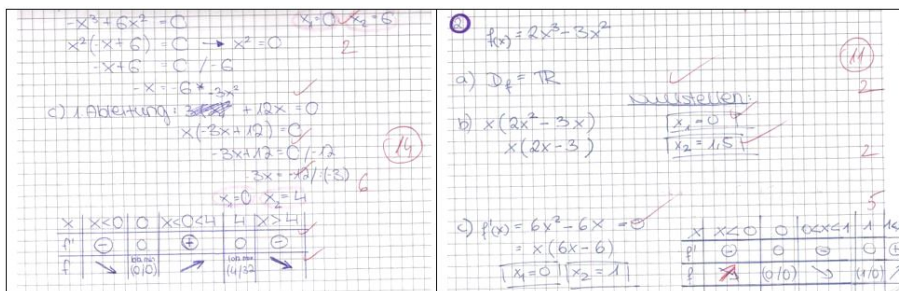


Figure 3: Pupils' solutions, table

- d) Everybody determined the y intercepts of the curve well.
- e) Those who did not draw the table well were naturally not able to do the sketching well. Among the 13 pupils with good tables 12 did the sketching well, one was still wrong (Figure 4).

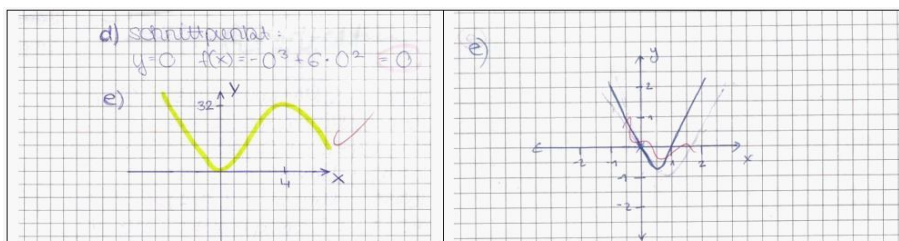


Figure 4: Pupils' results, sketching

**3rd task:**

Everybody was able to use the GC, they gave the command well, they got good zero points. However, five pupils got partly different results through calculations, still they did not examine the reasons for the differences (Figure 5).

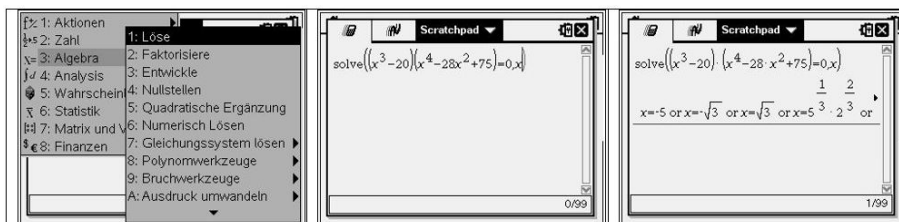


Figure 5: Calculation of zero points with GC

The following solution (Figure 6) is one of the above mentioned ones done in the other group, where the function given was the following:

$$f(x) = (x^4 - 12x^2 + 27)(x^2 - 4)$$

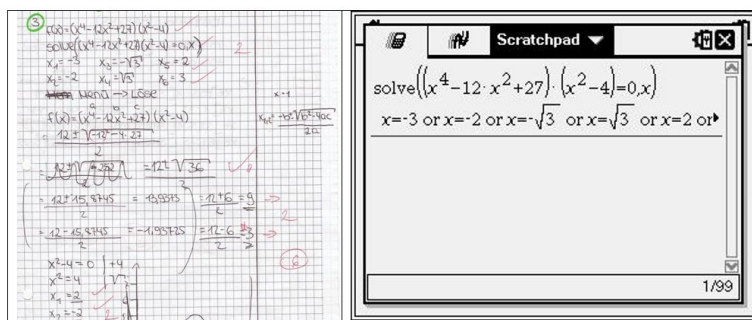


Figure 6: Partly good pupil's solution, done with GC

#### 4th task:

In task 4 the difficulties for some pupils were caused by the fact that the graphical calculator did not sketch the function in such interval that would have been appropriate for them. First the division of the axes should have been changed and then they would have been able to sketch it correctly. Those who did not do this, got partly wrong solutions. Still, 14 pupils did the sketching well which, in our opinion, was a very good result. The maxima, minima were correctly determined within the interval by all the 14 pupils who could use the calculator, whereas only six of them paid attention to examining the end points of the closed interval. Consequently, they could determine the intervals of increase and decrease perfectly.

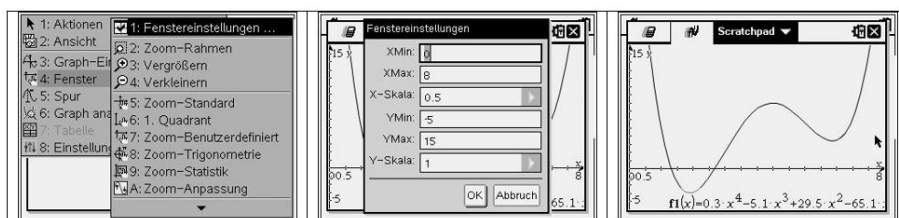


Figure 7: Calculation of zero points with graphical calculator

The results of tasks in the second test given in percentages were the following:

| 1.   | 2.  |      |      |     |      | 3.   | 4.   |      |      | Average |
|------|-----|------|------|-----|------|------|------|------|------|---------|
|      | a)  | b)   | c)   | d)  | e)   |      | a)   | b)   | c)   |         |
| 60.3 | 100 | 92.8 | 61.2 | 100 | 57.1 | 77.4 | 66.7 | 47.6 | 47.6 | 71.07%  |

Table 3: Results of second test

The averages of the second test were 2.37 (standard deviation: 1.21) and 2.58 (standard deviation: 1.62). The results of the parts done with and without GC were similar. In comparison with the previous results, we can say that pupils applied theory well in the exercises and they also used the calculator well mostly. All these results were better than the expected outcome. After both tests we discussed the tasks in details. In the parts where the pupils had questions the mathematical concept and the correct use of GC was explained again. At the end the pupils could separate the real properties of the functions from their graphical, visual properties. Because GC was used before and after this experiment, the pupils' skills and their confidence were growing during the school year.

## 5. Conclusions

The graphical calculator during teaching mathematics is still ambiguous. In German education system mathematics teachers allow and even encourage their students to use graphical calculators in education. On the one hand, they can provide support to do exercises that need a lot of calculations. It offers a comprehensive range of opportunities that need to be managed well. It has been supposed that it could improve student success in area such as introduction of differential calculus and at the same time could provide opportunities for visualization. The results from our experiment strongly indicated that graphical calculator can be an effective tool in this examined topic of mathematical education. The technology had a positive impact on students' successfulness and triggered their ability to improve their conceptual understanding. The students manipulated it with pleasure, though the proper use of each application required a lot of attention. They gave very positive feedback about parallel using of the "old" with the "new" method. Pupils appreciated that they could understand the mathematical meaning of concepts and at the same time they could use IT technology by calculations and representations.

On the other hand, teachers have to make sure that pupils are fully aware of the mathematical background as well. It was obvious that the students could understand the mathematical concepts with this "dual" method of teaching and learning. In spite of the fact that the technology can provide multiple representation of mathematical problems, such as in our experiment, the students have to be confident about the learned processes without the using graphical calculator where it is possible. While it is necessary to develop various ways to improve the students' skills in teaching and learning mathematics, taking into consideration our results, applying graphical calculators for mathematical problems can be a successful strategy in schools.

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## Appendix

|          |           | 10.a |       |      |       |       |      |       |      |      |       | 10.b |       |      |      |      |      |      |      |      |      |      |     |      |      |     |      |      |      |      |   |      |   |      |      |      |
|----------|-----------|------|-------|------|-------|-------|------|-------|------|------|-------|------|-------|------|------|------|------|------|------|------|------|------|-----|------|------|-----|------|------|------|------|---|------|---|------|------|------|
|          |           | ▼    |       |      |       |       | ▲    |       |      |      |       | ▼    |       |      |      |      | ▲    |      |      |      |      |      |     |      |      |     |      |      |      |      |   |      |   |      |      |      |
| 1.       | Satz      | 1    | 1     | 1    | 1     | 1     | 1    | 1     | 1    | 1    | 1     | 1    | 1     | 1    | 1    | 1    | 1    | 1    | 1    | 1    | 1    | 1    | 1   | 1    | 0    | 1   | 1    | 1    | 1    | 1    | 1 | 0    | 1 | 1    | 0    | 84,0 |
|          | Bew.      | 3    | 1     | 0    | 3     | 3     | 3    | 3     | 0    | 0    | 3     | 0    | 3     | 0    | 3    | 3    | 0    | 0    | 2    | 0    | 0    | 0    | 3   | 0    | 2    | 3   | 0    | 2    | 3    | 0    | 2 | 3    | 0 | 46,7 |      |      |
| 2.       | a)        | 3    | 3     | 3    | 3     | 3     | 3    | 3     | 1    | 3    | 0     | 3    | 3     | 3    | 3    | 2    | 0    | 3    | 1    | 1    | 0    | 1    | 3   | 2    | 2    | 3   | 0    | 69,3 |      |      |   |      |   |      |      |      |
|          | b)        | 1    | 1     | 1    | 1     | 0     | 1    | 1     | 1    | 1    | 1     | 1    | 0     | 0    | 1    | 1    | 0    | 1    | 1    | 0    | 1    | 1    | 0   | 1    | 1    | 1   | 1    | 1    | 1    | 1    | 0 | 68,0 |   |      |      |      |
|          | c)        | 3    | 2     | 3    | 1     | 3     | 0    | 2     | 2    | 1    | 2     | 0    | 3     | 0    | 1    | 3    | 1    | 2    | 2    | 1    | 2    | 2    | 3   | 0    | 1    | 3   | 0    | 53,3 |      |      |   |      |   |      |      |      |
|          | d)        | 2    | 2     | 2    | 2     | 2     | 1    | 2     | 2    | 2    | 0     | 2    | 0     | 1    | 1    | 1    | 2    | 2    | 0    | 2    | 2    | 2    | 2   | 2    | 1    | 1   | 2    | 0    | 72,0 |      |   |      |   |      |      |      |
|          | e)        | 1    | 0     | 1    | 1     | 0     | 1    | 1     | 1    | 1    | 1     | 0    | 0     | 1    | 1    | 0    | 1    | 1    | 0    | 1    | 1    | 1    | 1   | 1    | 1    | 1   | 1    | 1    | 0    | 68,0 |   |      |   |      |      |      |
| 3.       | a)        | 3    | 2     | 2    | 2     | 3     | 2    | 3     | 3    | 3    | 0     | 3    | 1     | 3    | 3    | 3    | 2    | 0    | 2    | 2    | 3    | 3    | 1   | 3    | 3    | 0   | 73,3 |      |      |      |   |      |   |      |      |      |
|          | b)        | 4    | 4     | 3    | 0     | 3     | 4    | 0     | 4    | 0    | 4     | 0    | 4     | 4    | 4    | 4    | 1    | 4    | 0    | 4    | 4    | 4    | 4   | 3    | 3    | 4   | 0    | 69,0 |      |      |   |      |   |      |      |      |
|          | c)        | 3    | 1,5   | 1,5  | 3     | 2,5   | 3    | 3     | 1    | 2    | 3     | 0    | 3     | 2    | 2    | 2,5  | 2    | 2,5  | 0,5  | 2,5  | 2    | 2,5  | 3   | 2    | 3    | 3   | 0,5  | 71,3 |      |      |   |      |   |      |      |      |
| 4.       | a)        | 4    | 4     | 4    | 4     | 4     | 4    | 4     | 4    | 4    | 4     | 4    | 4     | 4    | 4    | 4    | 4    | 4    | 4    | 4    | 4    | 4    | 4   | 4    | 4    | 4   | 4    | 4    | 4    | 4    | 4 | 4    | 4 | 95,0 |      |      |
|          | b)        | 4    | 4     | 4    | 4     | 4     | 4    | 4     | 2    | 4    | 4     | 4    | 2     | 4    | 4    | 2    | 4    | 4    | 2    | 4    | 4    | 2    | 4   | 4    | 4    | 4   | 4    | 4    | 4    | 4    | 4 | 4    | 4 | 1    | 78,0 |      |
| 5.       | a1)       | 3    | 3     | 3    | 3     | 3     | 3    | 3     | 3    | 3    | 3     | 3    | 3     | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 82,7 |      |      |
|          | a2)       | 2    | 2     | 2    | 2     | 2     | 2    | 2     | 2    | 2    | 2     | 2    | 2     | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2   | 2    | 2    | 2   | 2    | 2    | 2    | 2    | 2 | 2    | 2 | 80,0 |      |      |
|          | b1)       | 4    | 4     | 4    | 4     | 4     | 4    | 4     | 1    | 3    | 4     | 0    | 4     | 0    | 4    | 2    | 0    | 2    | 4    | 0    | 4    | 4    | 0   | 4    | 4    | 0   | 4    | 4    | 0    | 68,0 |   |      |   |      |      |      |
|          | b2)       | 2    | 2     | 2    | 2     | 2     | 2    | 2     | 2    | 2    | 2     | 2    | 2     | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2   | 2    | 2    | 2   | 2    | 2    | 2    | 2    | 2 | 2    | 2 | 60,0 |      |      |
| 6.       | a) Gl.    | 3    | 3     | 3    | 3     | 3     | 0    | 3     | 3    | 3    | 3     | 3    | 3     | 3    | 2    | 0    | 3    | 3    | 3    | 3    | 3    | 3    | 0   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 74,7 |      |      |
|          | b) Beschr | 2    | 2     | 2    | 2     | 2     | 2    | 2     | 2    | 2    | 2     | 2    | 2     | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2    | 2   | 2    | 2    | 2   | 2    | 2    | 2    | 2    | 2 | 2    | 2 | 76,0 |      |      |
|          | c) Dar    | 3    | 3     | 2    | 3     | 3     | 3    | 3     | 3    | 3    | 3     | 3    | 3     | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 85,3 |      |      |
| 7.       | a1) f     | 3    | 3     | 3    | 3     | 3     | 3    | 3     | 3    | 3    | 3     | 3    | 3     | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 92,0 |      |      |
|          | a2) f'    | 3    | 3     | 3    | 3     | 3     | 3    | 3     | 3    | 0    | 3     | 3    | 0     | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3    | 3   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 84,0 |      |      |
|          | b) Beschr | 3    | 3     | 3    | 3     | 3     | 3    | 3     | 3    | 0    | 3     | 3    | 2     | 3    | 3    | 3    | 0    | 1    | 3    | 0    | 3    | 2    | 3   | 3    | 3    | 3   | 3    | 3    | 3    | 3    | 3 | 3    | 3 | 74,7 |      |      |
| Total    |           | 60   | 53,5  | 53   | 53    | 56,5  | 52   | 55    | 45   | 41   | 59    | 1    | 58    | 33   | 56   | 50,5 | 21   | 48,5 | 41,5 | 29,5 | 51   | 42,5 | 60  | 30   | 54   | 60  | 10,5 | 44,6 |      |      |   |      |   |      |      |      |
| Part A % |           | 32   | 79,7  | 79,7 | 78,1  | 89,1  | 84,4 | 84    | 68,8 | 62,5 | 96,9  | 5,1  | 93,8  | 53,1 | 87,5 | 92,2 | 46,9 | 82,8 | 42,2 | 54,7 | 71,9 | 70,3 | 100 | 56,3 | 81,3 | 100 | 17,2 | 71,1 |      |      |   |      |   |      |      |      |
| Part B % |           | 28   | 100,0 | 96,4 | 100,0 | 100,0 | 89,3 | 100,0 | 82,1 | 75,0 | 100,0 | 0,0  | 100,0 | 57,1 | 100  | 75,0 | 21,4 | 78,6 | 100  | 42,9 | 100  | 71,4 | 100 | 42,9 | 100  | 100 | 17,9 | 78,0 |      |      |   |      |   |      |      |      |
| Total %  |           |      | 89,2  | 87,5 | 88,3  | 94,2  | 86,7 | 91,7  | 75,0 | 68,3 | 98,3  | 1,7  | 96,7  | 55,0 | 93,3 | 84,2 | 35,0 | 80,8 | 69,2 | 49,2 | 85,0 | 70,8 | 100 | 50,0 | 90,0 | 100 | 17,5 | 74,3 |      |      |   |      |   |      |      |      |
| Grade    |           |      | 1     | 1-   | 1-    | 1     | 1-   | 1     | 2    | 3+   | 1+    | 6    | 1+    | 4+   | 1    | 2+   | 5+   | 2+   | 2-   | 4    | 1-   | 2-   | 1+  | 4    | 1    | 1+  | 6    |      |      |      |   |      |   |      |      |      |