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# A general strong law of large numbers and applications to associated sequences and to extreme value theory*

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**Abstract**

The purpose of this paper is to establish a general strong law of large numbers (SLLN) for arbitrary sequences of random variables (rv's) based on the squared indice method and to provide applications to SLLN of associated sequences. This SLLN is compared to those based on the Hájek–Rényi type inequality. Nontrivial examples are given. An interesting issue that is related to extreme value theory (EVT) is handled here.

*Keywords:* Positive Dependence, Association, Negatively Associated, Hájek–Rényi Inequality, Max-Variance(r) Property, Strong Law of Large Numbers, Squared Indices Method, Extreme Value Theory, Hill's Estimator.

*MSC:* Primary 60F15, 62G20; Secondary 62G32, 62F12

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# 1. Introduction

In this paper, we present a general SLLN for arbitrary rv's and particularize it for associated sequences. In the recent decades both strong law of large numbers and central limit theorem for associated sequences have received and are still receiving huge interests since Lebowitz [13] and Newman [17] results under the strict stationarity assumption. The stationarity assumption was dropped by Birkel [3], who proved a version of a SLLN that can be interpreted as a generalized Kolmogorov's one. A recent account of such researches in this topic is available in [19]. Although many results are available for such sequences, there are still many open problems, especially regarding nonstationary sequences.

We intend to provide a more general SLLN for associated sequences as applications of a new general SLLN for arbitrary rv's. This new general SLLN is used to solve a remarkable issue of extreme value theory by using a pure probabilistic method.

Here is how this paper is organized. Since association is the central notion used here, we first make a quick reminder of it in Section 2. In Section 3, we make a round up of SLLN's available in the literature with the aim of comparing them to our findings. In Section 4, we state our general SLLN for arbitrary rv's and derive some classical cases. In Section 5, we give an application to EVT where the continuous Hill's estimator is studied by our method. The Section 6 concerns the conclusion and some perspectives are given. The paper is ended by the Appendix, where are postponed the proofs of Propositions 2 and 3 stated in Section 5.

To begin with, we give a short reminder of the concept of association.

# 2. A brief reminder of the concept of association

The notion of positive dependence for random variables was introduced by Lehmann (1966) (see [14]) in the bivariate case. Later this idea was extended to multivariate distributions by Esary, Proschan and Walkup (1967) (see [7]) under the name of association. The concept of association for rv's generalizes that of independence and seems to model a great variety of stochastic models. This property also arises in Physics, and is quoted under the name of FKG property (Fortuin, Kastelyn and Ginibre (1971), see [9]), in percolation theory and even in Finance (see [11]). The definite definition is given by Esary, Proschan and Walkup (1967) (see [7]) as follows.

**Definition 2.1.** A finite sequence of random variables $(X_1, \ldots, X_n)$ is associated if for any couple of real and coordinate-wise non-decreasing functions $f$ and $g$ defined on $\mathbb{R}^n$, we have

$$\mathrm{Cov}(f(X_1, \ldots, X_n),\ g(X_1, \ldots, X_n)) \geq 0$$

whenever the covariance exists. An infinite sequence of random variables is associated whenever all its finite subsequences are associated.

We have a few number of interesting properties to be found in ([19]): **(P1)** A sequence of independent rv's is associated. **(P2)** Partial sums of associated rv's are associated. **(P3)** Order statistics of independent rv's are associated. **(P4)** Non-decreasing functions and non-increasing functions of associated variables are associated. **(P5)** Let the sequence $Z_1, Z_2, \dots, Z_n$ be associated and let $(a_i)_{1\leq i\leq n}$ be positive numbers and $(b_i)_{1\leq i\leq n}$ real numbers. Then the rv's $a_i(Z_i - b_i)$ are associated.

As immediate other examples of associated sequences, we may cite Gaussian random vectors with nonnegatively correlated components (see [18]) and homogeneous Markov chains (see [4]).

The negative association was introduced by Joag-Dev and Proschan (1983) (see [12]) as follows

**Definition 2.2.** The variables $X_1, \dots, X_n$ are negatively associated if, for every pair of disjoint subsets nonempty $A$, $B$ of $\{1, \dots, n\}$, $A = \{i_1, \dots, i_m\}$, $B = \{i_{m+1}, \dots, i_n\}$ and for every pair of coordinatewise nondecreasing functions $f : \mathbb{R}^m \to \mathbb{R}$ and $g\colon \mathbb{R}^{n-m} \to \mathbb{R}$,

$$\operatorname{Cov}(f(X_i,\ i \in A),\ g(X_i,\ i \in B)) \leq 0 \tag{2.1}$$

whenever the covariance exists. An infinite collection is said to be negatively associated if every finite sub-collection is negatively associated.

*Remark* 2.3. For negatively associated sequences, we have (2.1), so the covariances are non-positive. This remark will be used in Subsubsection 4.1.2.

A usefull result of Newman (see [15]) on assocation, that is used in this paper, is the following

**Lemma 2.4** (Newman [15])**.** *Suppose that $X$ and $Y$ are two random variables with finite variance and, $f$ and $g$ are $\mathbb{C}^1$ complex valued functions on $\mathbb{R}^1$ with bounded derivatives $f'$ and $g'$. Then*

$$|\operatorname{Cov}(f(X), g(Y))| \leq ||f'||_\infty ||g'||_\infty \operatorname{Cov}(X, Y).$$

Here, we point out that strong laws of large numbers and, central limit theorem and invariance principle for associated rv's are available. Many of these results in that field are reviewed in [19]. Such studies go back to Lebowitz (1972) (see [13]) and Newman (1984) (see [17]). As Glivenko-classes for the empirical process for associated data, we may cite Yu (1993) (see [22]). We remind the results of such authors in this:

**Theorem 2.5** (Lebowitz [13] and Newman [17])**.** *Let $X_1, X_2, \dots$ be a strictly stationary sequence which is either associated or negatively associated, and let $T$ denote the usual shift transformation, defined so that*

$$T(f(X_{j_1}, \dots, X_{j_m})) = f(X_{j_1+1}, \dots, X_{j_m+1}).$$

*Then $T$ is ergodic (i.e., every $T$-invariant event in the $\sigma$-field generated by the $X_j$'s has probability* 0 *or* 1*) if and only if*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Cov}(X_1,X_j)=0. \tag{2.2}$$

*In particular, if* (2.2) *is valid, then for any $f$ such that $f(X_1)$ is $L_1$,*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}f(X_i)=\mathbb{E}\left(f(X_1)\right) \quad \textit{almost surely (a.s).}$$

Now we are going to state some classical SLLN's for arbitrary rv's in relation with Hájek–Rényi's scheme.

# 3. Strong laws of large numbers

For independent rv's, two approaches are mainly used to get SLLN's. A direct method using squared indice method seems to be the oldest one. Another one concerns the Kolmogorov's law based on the maximal inequality of the same name. Many SLLN's for dependent data are kinds of generalization of these two methods. Particularly, the second approach that has been developed to become the Hájek–Rényi's method (see [10]), seems to give the most general SLLN to handle dependent data. Since we will use such results to compare our findings to, we recall one of the most sophisticated forms of the Hájek–Rényi setting given by Tómács and Líbor (see [21]) denoted by (GCHR). These authors introduced a Hájek–Rényi's inequality for probabilities and, subsequently, got from it SLLN's for random sequences. They obtained first:

**Theorem 3.1.** *Let $r$ be a positive real number, $a_n$ be a sequence of nonnegative real numbers. Then the following two statements are equivalent.*
(i) *There exists $C>0$ such that for any $n\in\mathbb{N}$ and any $\varepsilon>0$*

$$\mathbb{P}\left(\max_{\ell\leq n}|S_\ell|\geq\varepsilon\right)\leq C\varepsilon^{-r}\sum_{\ell\leq n}a_\ell.$$

(ii) *There exists $C>0$ such that for any nondecreasing sequence $(b_n)_{n\in\mathbf{N}}$ of positive real numbers, for any $n\in\mathbb{N}$ and any $\varepsilon>0$*

$$\mathbb{P}\left(\max_{\ell\leq n}|S_\ell|b_\ell^{-1}\geq\varepsilon\right)\leq C\varepsilon^{-r}\sum_{\ell\leq n}a_\ell b_\ell^{-r}$$

*where $S_n=\sum_{i=1}^{n}X_i$ for all $n\in\mathbb{N}$.*

And next, they derived this SLLN from it.

**Theorem 3.2.** *Let $a_n$ and $b_n$ be non-negative sequences of real numbers and let $r > 0$. Suppose that $b_n$ is a positive non-decreasing, unbounded sequence of positive real numbers. Let us assume that*

$$\sum_n \frac{a_n}{b_n^r} < +\infty$$

*and there exists $C > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$*

$$\mathbb{P}\left(\max_{m\leq n} |S_m| \geq \varepsilon\right) \leq C\,\varepsilon^{-r} \sum_{m\leq n} a_m.$$

*Then*

$$\lim_{n\to+\infty} \frac{S_n}{b_n} = 0 \quad a.s.$$

For convenience, introduce these three notations. We say that a sequence of random variables $X_1$, $X_2, \ldots$ has the $\mathbb{P}$-*max-variance*$(r)$ property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$, for any $\lambda > 0$,

$$\mathbb{P}\left(\max\left(|S_1|, \ldots, |S_n|\right) \geq \lambda\right) \leq C\lambda^{-r}\,\mathbb{V}\mathrm{ar}(S_n).$$

It has the $\mathbb{V}$ar-*max-variance*$(r)$ property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$,

$$\mathbb{V}\mathrm{ar}(\max(|S_1|, \ldots, |S_n|))^{2/r} \leq C\,\mathbb{V}\mathrm{ar}(S_n)$$

and it has the $\mathbb{E}$-*max-variance*$(r)$ property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$,

$$\left(\mathbb{E}\left(\max(|S_1|, \ldots, |S_n|)\right)^2\right)^{2/r} \leq C\,\mathbb{V}\mathrm{ar}(S_n).$$

In the sequel we will say that *max-variance* property is satisfied if one of the three above max-variance properties holds.

Theorem 3.1 leads to these general laws.

**Proposition 1.** *Let $X_1$, $X_2, \ldots$ be a sequence of centered random variables. Let $(b_k)_{k\geq 1}$ be an increasing and nonbounded sequence of positive real numbers. Assume that*

$$\lim_{n\to+\infty}\sup \sum_{1\leq i\leq n} b_i^{-r}\,\mathrm{Cov}(X_i, S_n) < +\infty \tag{3.1}$$

*and the sequence has the $\mathbb{P}$-max-variance$(r)$ property, $r > 0$. Then $S_n/b_n \to 0$ a.s. as $n \to +\infty$.*

*If the sequence has the $\mathbb{V}$ar-max-variance$(2)$ property or the $\mathbb{E}$-max-variance$(2)$ property and if $\sum_{i\geq 1} b_i^{-2} \sum_{j\geq 1} \mathrm{Cov}(X_i, X_j) < +\infty$, then $S_n/b_n \to 0$ a.s. as $n \to +\infty$.*

*Remark* 3.3. Here, (3.1) is called the general condition of Hájek–Rényi ($GCHR$).

*Proof.* If the sequence has the $\mathbb{E}$-max-variance($r$) property, then there exists a constant $C > 0$ such that for any fixed $n \geq 1$, for any $\lambda > 0$, and for $r = 2$,

$$\begin{aligned}\mathbb{P}(\max(|S_1|, \ldots, |S_n|) \geq \lambda) &\leq \lambda^{-r}\, \mathbb{V}\text{ar}(\max(|S_1|, \ldots, |S_n|)) \\ &\leq \lambda^{-r} \mathbb{E}\left(\max(|S_1|, \ldots, |S_n|)\right)^2 \\ &\leq C\lambda^{-r}\, \mathbb{V}\text{ar}(S_n) = C\lambda^{-r} \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \text{Cov}(X_i, X_j) \right].\end{aligned}$$

The conclusion comes out by taking $a_i = \left[\sum_{j=1}^{n} \text{Cov}(X_i, X_j)\right] = \text{Cov}(X_i,\ S_n)$ in the Hájek–Rényi's Theorem 3.1 and applying Theorem 3.2. □

It is worth mentioning that the Hájek–Rényi's inequality is indeed very powerfull but, unfortunately, it works only if we have the max-variance property. For example, the $\mathbb{E}$-max property holds for strictly stationary and associated sequences (see [16]).

As to the squared indice method, it seems that it has not been sufficiently standed to provide general strong laws for dependent data. We aim at filling such a gap.

Indeed, in the next section, we provide a new general SLLN that inspired by the squared indice method. This SLLN will be showed to have interesting applications when comparing to the results of the present section.

# 4. Our results

In this section, we present a general SLLN based on the squared indice method and give different forms in specific types of dependent data including association with comparison with available results. The result will be used in Section 5 to establish the strong convergence for the continuous Hill's estimator with in the frame of EVT.

**Theorem 4.1.** *Let $X_1, X_2, \ldots$ be an arbitrary sequence of rv's, and let $(f_{i,n})_{i\geq 1, n\geq 1}$ be a sequence of measurable functions such that $\mathbb{V}\text{ar}[f_{i,n}(X_i)] < +\infty$, for $i \geq 1$ and $n \geq 1$. Let us suppose that for some $\delta, 0 < \delta < 3$,*

$$C_1 = \sup_{n\geq 1} \sup_{q\geq 1} \mathbb{V}\text{ar} \left( \frac{1}{q^{(3-\delta)/4}} \sum_{i=1}^{q} f_{i,n}(X_i) \right) < +\infty \tag{4.1}$$

*and that for some $\delta$, $0 < \delta < 3$,*

$$C_2 < +\infty, \tag{4.2}$$

*where $C_2$ is defined by*

$$\sup_{n>0} \sup_{k\geq 1} \sup_{q\ :\ q^2+1\leq k\leq (q+1)^2} \sup_{k\leq j\leq (q+1)^2} \mathbb{V}\text{ar} \left( \frac{1}{q^{(3-\delta)/2}} \sum_{i=1}^{j-q^2} f_{q^2+i,n}\left(X_{q^2+i}\right) \right).$$

*Then*

$$\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,n}(X_i)-\mathbb{E}\left(f_{i,n}(X_i)\right)\right)\to 0 \quad \textit{a.s. as} \quad n\to+\infty.$$

*Remark* 4.2. We say that the sequence $X_1, X_2, \ldots, X_n$ satisfies the (GCIP) whenever (4.1) and (4.2) hold.

*Proof.* It suffices to prove the announced results for $Y_i = f_{i,n}(X_i)$ and $\mathbb{E}(Y_i) = 0$, $i \geq 1$. Observe that omitting the subscript $n$ does not cause any ambiguity in the proof below. We have for any positive real number $\beta$,

$$\mathbb{P}\left(\left|\frac{1}{k}\sum_{i=1}^{k}Y_i\right|\geq k^{-\beta}\right)\leq \mathbb{P}\left(\left|\sum_{i=1}^{k}Y_i\right|\geq k^{1-\beta}\right)\leq \frac{1}{k^{2(1-\beta)}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{k}Y_i\right).$$

We apply this formula for $k = q^2$ and get for $0 < \delta < 3$,

$$\mathbb{P}\left(\left|\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i\right|\geq q^{-2\beta}\right)\leq \frac{1}{q^{4(1-\beta)}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{q^2}Y_i\right)$$

$$\leq \frac{1}{q^{1+\delta-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/2}}\sum_{i=1}^{q^2}Y_i\right)\leq \frac{C_1}{q^{1+\delta-4\beta}}.$$

Then we have for $0 < \beta < \delta/4$, $\sum_{q=1}^{+\infty}\mathbb{P}\left(\left|\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i\right| > q^{-2\beta}\right) < +\infty$. We conclude that

$$\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i \to 0 \quad \text{a.s. as} \quad q\to+\infty. \tag{4.3}$$

Now set $q^2 \leq k \leq (q+1)^2$ and $\epsilon_{k,q} = 0$ if $k = q^2$ and 1 otherwise. We have

$$\frac{1}{k}\sum_{i=1}^{k}Y_i-\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i=\frac{1}{k}\sum_{i=1}^{k}Y_i-\frac{1}{k}\sum_{i=1}^{q^2}Y_i+\frac{1}{k}\sum_{i=1}^{q^2}Y_i-\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i$$

$$=\frac{\epsilon_{k,q}}{k}\left(\sum_{i=1}^{k}Y_i-\sum_{i=1}^{q^2}Y_i\right)+\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i\left(\frac{q^2-k}{k}\right)$$

$$=\frac{\epsilon_{k,q}}{k}\left(\sum_{i=q^2+1}^{k}Y_i\right)+\frac{1}{q^2}\sum_{i=1}^{q^2}Y_i\left(\frac{q^2-k}{k}\right). \tag{4.4}$$

But $(q^2 - k)/k \to 0$ as $q \to +\infty$. This combined with (4.3) proves that the second term of (4.4) converges to zero a.s. It remains to handle the first term. For $0 < \delta < 3$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{k}\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k}Y_i\right|\geq k^{-\beta}\right)\leq \mathbb{P}\left(\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k}Y_i\right|\geq k^{1-\beta}\right)\\
&\leq \mathbb{P}\left(\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k}Y_i\right|\geq q^{2(1-\beta)}\right)\leq \frac{\epsilon_{k,q}}{q^{4-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{k}Y_i\right)\\
&\leq \frac{\epsilon_{k,q}}{q^{1+\delta-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/2}}\sum_{i=q^2+1}^{k}Y_i\right)\leq \frac{\epsilon_{k,\,q}C_2}{q^{1+\delta-4\beta}}.
\end{aligned}$$

Now for $0<\beta<\delta/4$, $\sum_{k=1}^{+\infty}\mathbb{P}\left(\epsilon_{k,q}\left|\sum_{i=q^2+1}^{k}Y_i\right|\geq k^{1-\beta}\right)<+\infty$. Then

$$\frac{\epsilon_{k,q}}{k}\left[\sum_{i=1}^{k}Y_i-\sum_{i=1}^{q^2}Y_i\right]\to 0 \quad \text{a.s. as} \quad q\to+\infty. \tag{4.5}$$

Now in view of (4.3), (4.4) and (4.5) and since $(q^2-k)/k\to 0$, we may conclude the proof. □

*Remark* 4.3. In most cases, conditions (4.1) and (4.2) are used for $\delta=1$, as it is the case for the independent and indentically distributed random variables. We will exhibit a situation in Proposition 2 that cannot be handled without using (4.1) and (4.2) for $\delta<1$.

## 4.1. Comparison and particular cases

Let us see how (GCIP), that is fulfilment of conditions (4.1) and (4.2), works in special cases. We have to compare our (GCIP) to (GCHR). But (GCHR) is used only when max-variance property is satisfied. We only consider the case where $X_1, X_2, \ldots$ are real and the $f_{i,n}$'s are identity functions.

### 4.1.1. Independence case.

By using Theorem 3.1, we observe that we have the $\mathbb{P}$-max-variance(2) property, that is the Kolmogorov's maximal inequality. By using the Hájek–Rényi's general condition, we have the strong law of large numbers of Kolmogorov: $S_n/n\to 0$ a.s. whenever

$$\sum_{n\geq 1}\mathbb{V}\mathrm{ar}(X_n)/n^2<+\infty.$$

To apply Theorem 4.1 here, we notice that the sequence of variances $\mathbb{V}\mathrm{ar}(S_n)$ is non-decreasing in $n$. Then (4.1) and (4.2) are implied by, for some $0<\nu_1$ and

$0 < \nu_2$,

$$\sup_{k\geq 1} \frac{1}{k^{1+\nu_1}} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) < +\infty \text{ and } \sup_{k\geq 1} \frac{1}{k^{2+\nu_2}} \sum_{i=k^2+1}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) < +\infty.$$

But, by observing that the latter is

$$k^{-(2+\nu_2)} \sum_{i=k^2}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) = k^{-(2+\nu_2)} \left[ \sum_{i=1}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) - \sum_{i=1}^{k^2} \mathbb{V}\mathrm{ar}(X_i) \right],$$

we conclude that the SLLN is implied by

$$\sup_{k\geq 1} \frac{1}{k^{1+\nu}} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) < +\infty, \tag{4.6}$$

some $\nu > 0$. In the independent case, one has the SLLN for $k^{-1} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) \to \sigma^2$. And the parameter $\nu$ in (4.6) is useless in that case. But the availability of the parameter $\nu$ is important for situations beyond the classical cases. As a first example, let us use the Kolmogorov's Theorem and construct a probability space holding a sequence of independent centered rv's $X_1, X_2, \ldots$ with $\mathbb{E}X_n^2 = n^{1/3}$. But (4.6) does not hold for $\nu = 0$ since

$$\frac{1}{n} \sum_{i=1}^{n} i^{1/3} \geq \frac{1}{n} \int_1^n x^{1/3} dx \geq \frac{3}{4} \left( n^{1/3} - 1 \right) \to +\infty, \quad \text{as} \quad n \to +\infty$$

while (GCHR) entails the SLLN.

We will consider in proposition 2 below an important other example which cannot be concluded unless we use a positive value of $\nu$. Now, if we may take $\nu = 1/3$, we have that $n^{-(1+\nu)} \sum_{i=1}^{n} i^{1/3}$ is bounded and our Theorem also ensures the SLLN.

Now if the sequence is second order stationary, then (4.1) and (4.2) are both valid. Also, if the variances are bounded by a common constant $C_0$, both (4.1) and (4.2) are valid.

#### 4.1.2. Pairwise negatively dependent variables.

In that case, we may drop the covariances in (GCIP) and then (4.1) and (4.2) lead to (4.6) as a general condition for the validity of the SLLN in the independent case. As to (GCHR), we don't have any information whether or not the max-variance property holds.

#### 4.1.3. Associated sequences

Here $\mathbb{V}\mathrm{ar}(S_n)$ is non-decreasing in $n$ and (GCIP) becomes for $\nu = (1-\delta)/2 \geq 0$ with $0 < \delta < 1$

$$\sup_{q\geq 1} \frac{1}{q^{1+\nu}} \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{q} X_i\right) < +\infty \tag{4.7}$$

and

$$\sup_{q\geq 1} \frac{1}{q^{2(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{(q+1)^2} X_i\right) < +\infty. \tag{4.8}$$

If the sequence is second order stationary, then (4.7) implies (4.8), since

$$\frac{1}{q^{2(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{(q+1)^2} X_i\right) = \frac{(2q+1)^{1+\nu}}{q^{2(1+\nu)}} \left[\frac{1}{(2q+1)^{1+\nu}} \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1} X_i\right)\right]$$
$$\sim \frac{2}{q^{(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\frac{1}{k^{(1+\nu)/2}} \sum_{i=1}^{k} X_i\right),$$

for $k = 2q+1$. And (4.7) may be witten as

$$\sup_{q\geq 1} \frac{1}{q^{\nu}} \left[\mathbb{V}\mathrm{ar}(X_1) + \frac{2}{q} \sum_{i=2}^{q} (q-i+1) \operatorname{Cov}(X_1, X_i)\right] < +\infty. \tag{4.9}$$

This is our general condition under which SLLN holds for second order stationary associated sequence. Then, by the Kronecker lemma, we have the SLLN if

$$\sigma^2 = \mathbb{V}\mathrm{ar}(X_1) + 2\sum_{i=2}^{+\infty} \operatorname{Cov}(X_1, X_i) < +\infty. \tag{4.10}$$

Condition (4.10) is obtained by Newman [16]. Clearly, by the Cesàro lemma, (4.10) implies

$$\lim_{q\to+\infty} \frac{1}{q} \sum_{i=1}^{q} \operatorname{Cov}(X_1, X_i) \to 0.$$

And, in fact, the latter is a necessary condition of strong law of large numbers as proved in Theorem 7 in [17], from the original result of Lebowitz (see [13]).

The reader may find a larger review on this subject in [19]. Our result seems more powerful since we may still have the strong law of large numbers even if $\sigma^2 = +\infty$.

We only need to check condition (4.9). We will comment this again after Proposition 2.

For strictly stationary associated sequences with finite variance, we have the $\mathbb{E}$-max-variance(2) property (see [16]). Then (GCHR) may be used. It becomes

$$\limsup_{n} \sum_{i=1}^{n} \frac{1}{i^2} \operatorname{Cov}(X_i, S_n) < +\infty, \tag{4.11}$$

which is equivalent to

$$\limsup_n \left[ \sum_{i=1}^{n} \frac{\mathbb{V}\mathrm{ar}(X_i)}{i^2} + \sum_{j=2}^{n} \left( \sum_{i=1}^{n-j+1} \frac{1}{i^2} + \sum_{i=j}^{n} \frac{1}{i^2} \right) \mathrm{Cov}(X_1, X_j) \right] < +\infty$$

and reduces to

$$\sum_{j=2}^{+\infty} \mathrm{Cov}(X_1, X_j) < +\infty.$$

We then see that (GCHR) gives weaker results than ours. Indeed, in our formula (4.9), we did not require that $\frac{2}{q}\sum_{i=2}^{q}(q-i+1)\,\mathrm{Cov}(X_1, X_i)$ is bounded. It may be allowed to go to infinity at a slower convergence rate than $q^{-\nu}$. Then our condition (4.9) besides being more general, applies to any associated sequences and is significantly better than the (GCHR) for strictly stationary sequences.

Nevertheless, for (4.11), it is itself more powerfull than Theorem 6.3.6 and Corollary 6.3.7 in [19], due to the use of Theorem 3.1 and Proposition 1, of Tómács and Líbor (see [21]). Such a result is also obtained by Yu (1993) (see [22]) for the strong convergence of empirical distribution function for associated sequence with identical and continuous distribution.

Birkel (see [3]) used direct computations on the convariance structure for associated variables and got the following condition

$$\limsup_n \sum_{i=1}^{n} \frac{1}{i^2} \mathrm{Cov}(X_i, S_i) < +\infty$$

for SLLN for associated variables.

Now, to sum up, the comparison between (GCIP) and (GCHR) is as follows:

1. For independent case the two conditions are equivalent.

2. In negatively associated case, the form of (GCIP) for independent case remains valid. And we have no information whether the max-variance property holds to be able to apply (GCHR).

3. For association with strictly stationary of sequences, (GCIP) gives a better condition than (GCHR).

4. For association with no information on stationarity, so (GCHR) cannot be applied unless a max-variance property is proved. Our condition still works and is the same as for the stationary associated sequences in point 3.

5. For arbitrary sequences with finite variances, point 4 may be recontacted.

In conclusion, our method effectively brings a significant contribution to SLLN for associated random variables. And we are going to apply it to an associated sequence in the extreme value theory fields.

# 5. Applications

## 5.1. Application to extreme value theory

The EVT offers us the opportunuity to directly apply our general conditions (4.1) and (4.2) to a sum of dependent and non-stationary random variables and to show how to proceed in such a case.

We already emphasized the importance of the parameter $\nu = (1-\delta)/2$ in (GCIP). In the example we are going to treat, we will see that a conclusion cannot be achieved with $\nu = 0$.

Let $E_1, E_2, \ldots$ be an infinite sequence of independent standard exponential random variables, $f(j)$ is an increasing function of the integer $j \geq 0$ with $f(0) = 0$ and $\gamma > 0$ a real parameter. Define the following sequences of random variables

$$W_k = \sum_{j=1}^{k-1} f(j) \left[ \exp\left( -\gamma \sum_{h=j+1}^{k-1} E_h/h \right) - \exp\left( -\gamma \sum_{h=j}^{k-1} E_h/h \right) \right], \ k \geq 1. \tag{5.1}$$

The characterization of the asymptotic behavior of (5.1) has important applications and consequences in two important fields: the extreme value theory in statistics and the central limit theorem issue for sum of non stationary associated random variables. Let us highlight each of these points.

On one side, let $X, X_1, X_2, \ldots$ be independent and identically random variables in Weibull extremal domain of parameter $\gamma > 0$ such that $X > 0$ and let $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the $n \geq 1$ observations. The distribution function $G$ of $Y = \log X$ has a finite upper endpoint $y_0$ and admits the following representation:

$$y_0 - G^{-1}(1-u) = cu^{1/\gamma}(1 + p(u)) \exp\left( \int_u^1 t^{-1} b(t) dt \right), \ u \in (0,1)$$

where $c$ is a constant and, $p(u)$ and $b(u)$ are functions of $u \in (0,1)$ such that $(p(u), b(u)) \to 0$ as $u \to 0$. This is called a representation of a sequence of random variables in the Weibull domain of attraction.

To stay simple, suppose that $p(u) = b(u) = 0$ for all $u \in (0,1)$ consider the simplest case

$$y_0 - G^{-1}(1-u) = u^{\gamma}, \ u \in (0,1). \tag{5.2}$$

The so-called Hill's statistic, based on the identity function $id(x) = x$ and the $k$ largest values with $1 \leq k \leq n$,

$$T_n(id) = \frac{1}{id(k)} \sum_{j=1}^{k} id(j) \left( \log X_{n-j+1,n} - \log X_{n-j,n} \right)$$

is an estimator of $\gamma$ in the sense that

$$\frac{T_n(id)}{(y_0 - \log X_{n-k,n})} \to^{\mathbb{P}} (\gamma+1)^{-1},$$

as $n \to +\infty$. When we replace the identity function with an increasing function $f(j)$ of the integer $j \geq 0$ with $f(0) = 0$, we get the functional Hill's estimator defined as

$$T_n(f) = \frac{1}{f(k)} \sum_{j=1}^{k} f(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right)$$

introduced by Dème E., Lo G.S. and Diop, A. (2012) (see [5]). From this processus is derived the Diop and Lo (2006) (see [6]) generalization of Hill's statistic. We are going to highlight that $f(k)T_n(f)/(y_0 - \log X_{n-k,n})$ is of the form of (5.1) when (5.2) holds. We have to use two representations. The Rényi's representation allows to find independent standard uniform random variables $U_1, U_2, \ldots$ such that the following equalities in distribution hold

$$\{\log Y_j,\ j \geq 1\} =_d \{G^{-1}(1 - U_j),\ j \geq 1\}$$

and

$$\{\{\log X_{n-j+1,n},\ 1 \leq j \leq n\},\ n \geq 1\} =_d \left\{\{G^{-1}(1 - U_{j,n}),\ 1 \leq j \leq n\},\ n \geq 1\right\}.$$

Next, by the Malmquist representation (see ([20]), p. 336), we have for each $n \geq 1$, the following equality in distribution holds

$$\{j^{-1} \log(U_{j+1,n}/U_{j,n}),\ 1 \leq j \leq n\} =_d \{E_j^{(n)},\ 1 \leq j \leq n\},$$

where $E_j^{(n)}$, $1 \leq j \leq n$, are independent exponential random variables. We apply these two tools to get that for each fixed $n$ and $k = k(n)$

$$\frac{T_n(f)}{(y_0 - \log X_{n-k,n})} =_d W_{k(n)}. \tag{5.3}$$

For an arbitrary element of the Weibull extremal domain of attraction, it may be easily showed that $f(k)T_n(f)/(y_0 - \log X_{n-k,n})$ also behaves as (5.1) if some extra conditions are imposed of the auxiliary functions $p$ and $b$. Hence a complete characterization of the asymptotic behavior of (5.1) provides asymptotic laws in extreme value theory.

On another side, easy algebra leads to

$$W_k = f(k-1) + \sum_{j=1}^{k-1} \Delta f(j) \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right),$$

where $\Delta f(j) = f(j) - f(j-1)$, $j \geq 1$. We consider

$$W_k^* = W_k - \mathbb{E}(W_k) = \sum_{j=1}^{k-1} \Delta f(j) \left[\exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right) - \mathbb{E} \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right)\right].$$

This is a sum of non stationary dependent random variables. In fact the $rv$'s

$$\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)-\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]$$

are associated.

Now, we are going to apply our general conditions to (5.4), defined below

$$S_k^*=\sum_{j=1}^{k-1}\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)-\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]\alpha(k),\qquad(5.4)$$

where $\alpha(k)$ is a sequence of positive real numbers. Next, we will particularize the result for $f(j)=j^{\tau}$, $\tau>0$. Our results depend on computation techniques developed in [8]. Here are our results:

**Proposition 2.** *Suppose that, for $L$ and $q$ large enough such that $L\leq q^2$, the following conditions hold for some $\delta$, $0<\delta<3$.*

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{2\gamma+1+\nu}}\sum_{j=L}^{k-1}\Delta^2 f(j)j^{2\gamma}<+\infty,\qquad(5.5)$$

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{1+\nu}}\sum_{j=L+1}^{k-1}\left[\sum_{i=L}^{j-1}\Delta f(i)\right]\Delta f(j)\frac{1}{j}<+\infty,\qquad(5.6)$$

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{1+\nu}}\sum_{L\leq j\leq k-1}\Delta f(j)/j<+\infty,\qquad(5.7)$$

$$\sup_{k\geq 1}\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\left(\frac{q^2+i}{k}\right)^{2\gamma}<+\infty\qquad(5.8)$$

*and*

$$\sup_{k\geq 1}\sup_{(q^2+1)\leq k\leq(q+1)^2}\frac{\alpha^2(k)}{q^{(3-\delta)}}\sum_{j=2}^{2q+1}\left[\sum_{i=1}^{j-1}\Delta f(q^2+i)\right]\Delta f(q^2+j)\frac{1}{q^2+j}<+\infty.\qquad(5.9)$$

*Then*

$$\frac{S_k^*}{k}\to 0\quad a.s.$$

*Further, if*

$$\mu_k=\sum_{j=1}^{k-1}\alpha(k)\Delta f\,(j)\,\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\to\mu,$$

*where $\mu$ is a finite, then*

$$k^{-1}\sum_{j=1}^{k-1}\alpha(k)\Delta f\,(j)\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\to\mu\quad a.s.$$

**Proposition 3.** *For* $f(j) = j^\tau$, *if* (5.5), (5.6), (5.7), (5.8) *and* (5.9) *hold,* $\alpha(k) = 1/k^{\tau-1}$ *and if* $\mu = \tau/(\tau+\gamma)$. *Then*

$$\frac{1}{k^\tau}\sum_{j=1}^{k-1}(j^\tau - (j-1)^\tau)\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right) \to \frac{\tau}{\gamma+1} \quad a.s.\ as \quad k\to+\infty.$$

*Remark* 5.1. Since these results are only based on moments, the a.s. convergence remains true for $T_n(f)/(y_0 - \log X_{n-k,n})$ in vertue of (5.3). We get under the model that

$$\frac{T_n(f)}{(y_0 - \log X_{n-k,n})} \to \frac{\tau}{\gamma+1} \text{ a.s. as } n\to+\infty \text{ and } k=k(n)\to+\infty \text{ and } k/n\to 0$$

under the assumptions (5.5), (5.6), (5.7), (5.8) and (5.9), in the general case.

*Remark* 5.2. This strong law may be easily checked by Monte Carlo simulations. For example, consider $\gamma = 2$ and $\tau = 1$. We observe the following errors corresponding to the values of 50, 75 and 100 of $k$: 0.358, 0.321 and 0.3332. This shows the good performance of this strong law for the particular values $\gamma = 2$ and $\tau = 1$.

#### 5.1.1. Proofs

Both proofs of the two propositions are postponed in the Appendix.

# 6. Conclusion and perspectives

We have established a general SLLN and applied it to associated variables. Comparison with SLLN's derived from the Hájek-Rényi inequality proved that this SLLN is not trivial. We have also used it to find the strong convergence of statistical estimators under non-stationary associated samples in EVT.

It seems that it has promising applications in non-parametric statistic, when dealing with the strong convergence of the empirical process and the non-parametric density estimator for a stationary sequence with an arbitrary parent distribution function.

# 7. Appendix

## 7.1. Proofs of Proposition 2 and Proposition 3

#### 7.1.1. Assumptions

We have to show that the assumptions of Proposition 2 entail the general condition (GCIP). We first remind that

$$S_k^* = \sum_{j=1}^{k-1}\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right) - \mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]\alpha(k)$$

that we write as

$$S_k^* = \sum_{j=1}^{k-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right), \tag{7.1}$$

where $S_{j,k} = \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right)$ and $s_{j,k} = \mathbb{E}\exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right)$. Next, we are going to check (4.1) and (4.2) for this sum of random variables. Fix $\delta$, $0 < \delta < 3$. Let us split (7.1) into

$$S_k^* = \sum_{j=1}^{L-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right) + \sum_{j=L}^{k-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right) =: S_L^1 + S_L^2.$$

Then for $\nu = (1-\delta)/2$ with $0 < \delta < 1$,

$$\begin{aligned}\frac{1}{k^{1+\nu}}\,\mathbb{V}\mathrm{ar}(S_k^*) &= \frac{1}{k^{1+\nu}}\mathbb{V}ar\left(S_L^1\right) + \frac{1}{k^{1+\nu}}\,\mathbb{V}\mathrm{ar}\left(S_L^2\right) + \frac{2}{k^{1+\nu}}\,\mathrm{Cov}\left(S_L^1, S_L^2\right)\\ &=: A_k + B_k + 2C_k.\end{aligned}$$

Let us treat each term in the above equality. Here, we use Formulas 18 and 21 in [8] and take $L$ large enough to ensure

$$\mathbb{V}\mathrm{ar}\left(S_{j,k}\right) = \left(\frac{j}{k-1}\right)^{2\gamma} V(1,j)V(2,j), \tag{7.2}$$

with

$$|V(1,j)| = 1 + O(j^{-1}) \text{ and } 0 \leq V(2,j) \leq \frac{2\gamma^2\,|a_1(\in)|}{j}$$

and

$$\mathrm{Cov}(S_{j,k}, S_{j+\ell,k}) = \mathbb{V}\mathrm{ar}\left(S_{j+\ell,k}\right)\left(\frac{j}{j+\ell-1}\right)^{\gamma}(1 + O(j^{-1})).$$

We suppose that $L$ is large enough so that $|V(1,j)| \leq 1/2$, for $j \geq L$.

First we see that

$$A_k \to 0, \quad \text{as} \quad k \to +\infty, \tag{7.3}$$

since $\mathbb{V}\mathrm{ar}(S_L^1)$ is let constant with $L$. Next, split $B_k$ into

$$\begin{aligned}B_k &= \frac{1}{k^{1+\nu}} \sum_{j=L}^{k-1} \alpha^2(k)\Delta^2 f(j)\,\mathbb{V}\mathrm{ar}\left(S_{j,k} - s_{j,k}\right)\\ &+ \frac{1}{k^{1+\nu}} \sum_{L \leq i \neq j \leq k-1} \alpha^2(k)\Delta f(j)\Delta f(i)\,\mathrm{Cov}\left(S_{i,k}, S_{j,k}\right)\\ &=: B_{k,1} + B_{k,2}.\end{aligned}$$

By (7.2) we get

$$B_{k,1} = \frac{1}{k^{1+\nu}} \sum_{j=L}^{k-1} \alpha^2(k)\Delta^2 f(j)\, \mathbb{V}\mathrm{ar}\,(S_{j,k}) \leq (1/2)\frac{\alpha^2(k)}{k^{2\gamma+1+\nu}} \sum_{j=L}^{k-1} \Delta^2 f(j) j^{2\gamma}. \quad (7.4)$$

Now let us turn to the term $B_{k,2}$. Let us remark that the rv$'$s $S_{j,k}$ are non increasing functions of independent rv's $E_j$. So they are associated. We then use the Lemma 3 of Newman [15] stated in Lemma 2.4 to get

$$\left|\mathrm{Cov}\left(\exp\left(-\gamma\sum_{h=i}^{k-1} E_h/h\right),\ \exp\left(-\gamma\sum_{h=j}^{k-1} E_h/h\right)\right)\right|$$
$$\leq \mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right),$$

where we use the one-value bound of $\exp(-x)$. For $i \leq j$,

$$\mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right) = \mathbb{V}\mathrm{ar}\left(\gamma\sum_{h=j}^{k-1} E_h/h\right) = \gamma^2\sum_{h=j}^{k-1} h^{-2} \leq \frac{\gamma^2}{j}, \quad (7.5)$$

the latter inequality is directly obtained by comparing $\sum_{h=j}^{k-1} h^{-2}$ and $\int_j^k x^{-2}dx$. We get

$$\begin{aligned}
|B_{k,2}| &\leq \frac{1}{k^{1+\nu}} \sum_{L\leq i\neq j\leq k} \alpha^2(k)\Delta f(j)\Delta f(i)\,\mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right) \\
&\leq \frac{2\gamma^2}{k^{1+\nu}} \sum_{L\leq i<j\leq k} \alpha^2(k)\Delta f(j)\Delta f(i)/j \\
&= \frac{2\gamma^2}{k^{1+\nu}}\alpha^2(k) \sum_{j=L+1}^{k-1} \left[\sum_{i=L}^{j-1} \Delta f(i)\right] \Delta f(j)\frac{1}{j}. \quad (7.6)
\end{aligned}$$

Finally, by using the techniques of (7.5) and (7.6), we get

$$\begin{aligned}
C_k &= \sum_{1\leq i\leq L-1}\ \sum_{L\leq j\leq k-1} \alpha^2(k)\Delta f(i)\Delta f(j)\,\mathrm{Cov}(S_{i,k}, S_{j,k}) \\
&\leq \frac{\alpha^2(k)\gamma^2}{k^{1+\nu}} \sum_{L\leq j\leq k-1} \left[\sum_{1\leq i\leq L-1} \Delta f(i)\right] \Delta f(j)/j, \quad (7.7)
\end{aligned}$$

where $\left[\sum_{1\leq i\leq L-1} \Delta f(i)\right]$ is a constant. By putting together (7.3), (7.4), (7.6) and (7.7), we get that assumptions (5.5), (5.6) and (5.7) entail (4.1) in Theorem 4.1.

We are going to check for (4.2) now. We already noticed that the rv's $\alpha(k)\Delta f(q^2+i)(S_{k,q^2+i}-s_{k,q^2+i})$ are associated and partial sums of associated rv's have non decreasing variances. Then for $j \leq 2q+1$, we have

$$\begin{aligned}&\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{j}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right)\\ &\leq \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right).\end{aligned}$$

And (4.2) becomes

$$\sup_{k\geq 1}\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right). \tag{7.8}$$

We fix $q$ but large enough to ensure $q^2 \geq L$, where $L$ is introduced in (7.2). So (7.8) is bounded by

$$\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right).$$

Now, we only have to show that

$$D=\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right)$$

is bounded for $q^2 \geq L$. Let us split term in the brackets into

$$\begin{aligned}D&=\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\,\mathbb{V}\mathrm{ar}\left(S_{k,q^2+i}\right)\\ &\quad+\frac{1}{q^{(3-\delta)}}\sum_{1\leq i\neq j\leq 2q+1}\alpha^2(k)\Delta f(q^2+i)\Delta f(q^2+j)\,\mathrm{Cov}\left(S_{k,q^2+i},S_{k,q^2+j}\right)\\ &=: D_1+D_2.\end{aligned}$$

We have, by (7.2),

$$\begin{aligned}D_1&=\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\,\mathbb{V}\mathrm{ar}\left(S_{k,q^2+i}\right)\\ &\leq (1/2)\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\left(\frac{q^2+i}{k}\right)^{2\gamma}.\end{aligned} \tag{7.9}$$

Now, we handle $D_2$. We use again the techniques that lead to (7.6) based on the Newman's Lemma to get, for $i \leq j$,

$$|D_2| \leq \frac{1}{q^{(3-\delta)}} \sum_{1 \leq i \neq j \leq 2q+1} \alpha^2(k) \Delta f(q^2+i) \Delta f(q^2+j) \, \mathbb{V}\mathrm{ar}\left(\gamma \sum_{h=q^2+j}^{2q+1} E_h/h\right).$$

We remind, as in (7.5), that

$$\mathbb{V}\mathrm{ar}\left(\gamma \sum_{h=q^2+j}^{2q+1} E_h/h\right) \leq \gamma^2/(q^2+j)$$

and then

$$\begin{aligned} |D_2| &\leq \frac{2\gamma^2}{q^{(3-\delta)}} \alpha^2(k) \sum_{1 \leq i < j \leq 2q+1} \Delta f(q^2+i) \Delta f(q^2+j) \frac{1}{q^2+j} \\ &= \frac{2\gamma^2}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=2}^{2q+1} \left[\sum_{i=1}^{j-1} \Delta f(q^2+i)\right] \Delta f(q^2+j) \frac{1}{q^2+j}. \end{aligned} \tag{7.10}$$

By putting together (7.9) and (7.10), we get that assumptions (5.8) and (5.9) entail (4.2) in Theorem 4.1. We may conclude that the strong law of large numbers holds for $S_k^*$.

### 7.1.2. Special case for $f(j) = j^\tau$

We are going to check the conditions (5.5), (5.6), (5.7), (5.8) and (5.9) for the special function $f(j) = j^\tau$, $\tau > 0$. We fix $L$ as indicated, consider $q \geq L$ and work with $k \geq q^2+1$. We notice that $\Delta f(j)$ is equivalent to $\tau j^{\tau-1}$ and $\Delta f(q^2+j)$ is uniformly equivalent to $\tau j^{\tau-1}$ uniformly in $j \geq L$. Here $\alpha(k) = k^{-(\tau-1)}$. Then (5.5) holds when

$$\sup_{k \geq L} \frac{\tau^2}{k^{2\gamma+2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{2\gamma+2\tau-2}$$

is bounded. But if $2\gamma+2\tau-1=0$, we get

$$\frac{1}{k^\nu} \sum_{j=L}^{k-1} j^{-1} \sim k^{-\nu} \log k \to 0$$

and for $2\gamma+2\tau-1 \neq 0$, we get

$$\frac{1}{k^{2\gamma+2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{2\gamma+2\tau-2} \sim k^{-\nu}(2\gamma+2\tau-1)^{-1}$$

and (5.5) holds. (5.6) holds with boundedness of

$$\sup_{k\geq L} \frac{1}{k^{2\tau-1+\nu}} \sum_{j=L+1}^{k-1} j^{2\tau-2}$$

which is, for $2\tau - 1 \neq 0$

$$\frac{1}{2\tau-1} k^{-\nu} \to 0,$$

and for $2\tau = 1$

$$k^{-\nu} \ln k \to 0.$$

Next (5.7) is equivalent to the boundedness of

$$\frac{1}{k^{2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{\tau-2},$$

which is equivalent to the boundedness of $k^{-(\tau+\nu)} \log k$, for $\tau - 1 = 0$ and to that of $k^{-(\tau+\nu)}$ for $\tau - 1 \neq 0$. Let us now handle (5.8) which is equivalent to the boundedness of

$$\frac{1}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=2}^{2q+1} \left[ \sum_{i=1}^{j-1} \Delta f(q^2+i) \right] \Delta f(q^2+j) \frac{1}{q^2+j}$$
$$\leq \frac{1}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=1}^{2q+1} \left[ \sum_{i=1}^{j-1} \Delta f(q^2+i) \right] \Delta f(q^2+j) \frac{1}{q^2+j},$$

for enough large $q$. We have to establish that

$$\sup_{k\geq 1} \sup_{(q^2+1)\leq k\leq (q+1)^2} \frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \sum_{j=1}^{2q+1} (q^2+j)^{2\gamma+2\tau-2} < +\infty.$$

If $2\gamma + 2\tau - 1 \neq 0$, then $\frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \sum_{j=1}^{k-(2k+1-q^2)} (q^2+j)^{2\gamma+2\tau-2}$ is bounded whenever

$$\frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \frac{k^{2\gamma+2\tau-1}}{2\gamma+2\tau-1} = \frac{1}{2\gamma+2\tau-1} (k/q^2) q^{-(1-\delta)}$$

is bounded. And if $2\gamma+2\tau-1 = 0$, $\sum_{j=1}^{k-(2k+1-q^2)} (q^2+j)^{2\gamma+2\tau-2}$ is bounded along with

$$\frac{k}{q^{(3-\delta)}} \log k \leq \left(k/q^2\right) q^{-(1-\delta)} \log k.$$

In both cases, $\left(k/q^2\right) q^{-(1-\delta)} \sim q^{-(1-\delta)} \to 0$ as $k$ (and $q$) goes to infinity. The proof is now complete.

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