Annales Mathematicae et Informaticae
45 (2015) pp. 107–110
http://ami.ektf.hu

A note on log-convexity of power means

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Submitted January 22, 2015 — Accepted September 29, 2015

Abstract

We point out results connected with the log-convexity of power means of two arguments.

1. Introduction

Let $M_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ $(p \neq 0)$, $M_0(a,b) = \sqrt{ab}$ denote the power mean (or Hölder mean, see [2]) of two arguments $a, b > 0$. Recently A. Bege, J. Bukor and J. T. Tóth [1] have given a proof of the fact that for $a \neq b$, the application $p \to M_p$ is log-convex for $p \leq 0$ and log-concave for $p \geq 0$. They also proved that it is also convex for $p \leq 0$. We note that this last result follows immediately from the well-known convexity theorem, which states that all log-convex functions are convex, too (see e.g. [2]). The proof of authors is based on an earlier paper by T. J. Mildorf (see [1]).

In what follows, we will show that this result is well-known in the literature, even in a more general setting. A new proof will be offered, too.

2. Notes and results

In 1948 H. Shniad [6] studied the more general means $M_t(a,\xi) = \left(\sum_{i=1}^n \xi_i a_i^t\right)^{1/t}$ $(t \neq 0)$, $M_0(a,\xi) = \prod_{i=1}^n a_i^{\xi_i}$, $M_{-\infty}(a,\xi) = \min\{a_i : i = 1, \dots\}$, $M_{+\infty}(a,\xi) = \max\{a_i : i = 1, \dots\}$; where $0 < a_i < a_{i+1}$ $(i = 1, \dots, n-1)$ are given positive real numbers, and $\xi_i (i = \overline{1,n})$ satisfy $\xi_i > 0$ and $\sum_{i=1}^n \xi_i = 1$.

Put $\Lambda(t) = \log M_t(a,\xi)$. Among other results, in [6] the following are proved:

Theorem 2.1.
(1) *If* $\xi_1 \geq \frac{1}{2}$ *then* $\Lambda(t)$ *is convex for all* $t < 0$.

(2) *If* $\xi_n \geq \frac{1}{2}$ *then* $\Lambda(t)$ *is concave for all* $t > 0$.

Clearly, when $n = 2$, in case of M_p one has $\xi_1 = \xi_2 = \frac{1}{2}$, so the result by Bege, Bukor and Tóth [1] follows by Theorem 2.1.

Another generalization of power mean of order two is offered by the Stolarsky means (see [7]) for $a, b > 0$ and $x, y \in \mathbb{R}$ define

$$D_{x,y}(a,b) = \begin{cases} \left[\frac{y(a^x-b^x)}{x(a^y-b^y)}\right]^{1/(x-y)}, & \text{if } xy(x-y) \neq 0, \\ \exp\left(-\frac{1}{x} + \frac{a^x \ln a - b^x \ln b}{a^x - b^x}\right), & \text{if } x = y \neq 0, \\ \left[\frac{a^x-b^x}{(\ln a - \ln b)}\right]^{1/x}, & \text{if } x \neq 0, y = 0, \\ \sqrt{ab}, & \text{if } x = y = 0. \end{cases}$$

The means $D_{x,y}$ are called sometimes as the difference means, or extended means.

Let $I_x(a,b) = (I(a^x, b^x))^{1/x}$, where $I(a,b)$ denotes the identic mean (see [2, 4]) defined by

$$I(a,b) = D_{1,1}(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad (a \neq b),$$
$$I(a,a) = a.$$

K. Stolarsky [7] proved also the following representation formula:

$$\log D_{x,y} = \frac{1}{y-x} \int_x^y \log I_t dt \text{ for } x \neq y.$$

Now, in 2001 the author [4] proved for the first time that the application $t \to \log I_t$ is convex for $t < 0$ and concave for $t > 0$.

This in turn implies immediately (see also [3]) the following fact:

Theorem 2.2.
(1) *If* $x > 0$ *and* $y > 0$, *then* $D_{x,y}$ *is log-concave in both* x *and* y.

(2) *If* $x < 0$ *and* $y < 0$, *then* $D_{x,y}$ *is log-convex in both* x *and* y.

Now, remark that

$$M_p(a,b) = D_{2p,p}(a,b),$$

so the log-convexity properties by H. Shniad are also particular cases of Theorem 2.2.

We note that an application of log-convexity of M_p is given in [5].

3. A new elementary proof

We may assume (by homogenity properties) that $b = 1$ and $a > 1$. Let $f(p) = \ln((a^p + 1)/2)/p$, and denote $x = a^p$. Then, as $x' = \frac{dx}{dp} = a^p \ln a = x \ln a$, from the identity $pf(p) = \ln(x + 1)/2$ we get by differentiation

$$f(p) + pf'(p) = \frac{x \ln a}{x + 1}. \tag{3.1}$$

By differentiating once again (3.1), we get

$$2f'(p) + pf''(p) = \frac{(x \ln^2 a)(x + 1) - x^2 \ln^2 a}{(x + 1)^2},$$

which implies, by definition of $f(p)$ and relation (3.1):

$$\begin{aligned} p^3 f''(p) &= \frac{(x \ln^2 x)(x + 1) - x^2 \ln^2 x}{(x + 1)^2} - \frac{2}{x + 1}\left[x \ln x - (x + 1) \ln\left(\frac{x + 1}{2}\right)\right] \\ &= \frac{x \ln^2 x + 2(x + 1)^2 \ln(x + 1) - 2x(x + 1) \ln x}{(x + 1)^2}, \end{aligned}$$

after some elementary computations, which we omit here. Put

$$g(x) = x \ln^2 x + 2(x + 1)^2 \ln\left(\frac{x + 1}{2}\right) - 2x(x + 1) \ln x.$$

One has successively:

$$\begin{aligned} g'(x) &= ln^2 x + 4(x + 1) \ln\left(\frac{x + 1}{2}\right) - 4x \ln x, \\ g''(x) &= \frac{2 \ln x}{x} + 4 \ln\left(\frac{x + 1}{2}\right) - 4 \ln x, \\ g'''(x) &= 2\left[\frac{1 - \ln x}{x^2} - \frac{2}{x(x + 1)}\right] = \frac{-2}{x^2(x + 1)}[x - 1 + (x + 1) \ln x]. \end{aligned}$$

Now, remark that for $x > 1$, clearly $g'''(x) < 0$, so $g''(x)$ is strictly decreasing, implying $g''(x) < g''(1) = 0$. Thus $g'(x) < g'(1) = 0$, giving $g(x) < g(1) = 0$. Finally, one gets $f''(p) < 0$, which shows that for $x > 1$ the function $f(p)$ is strictly concave function of p. As $x = a^p$ with $a > 1$, this happend only when $p > 0$.

For $x < 1$, remark that $x - 1 < 0$ and $\ln x < 0$, so $g'''(x) > 0$, and all above procedure may be repeted. This shows that $f(p)$ is a strictly convex function of p for $p < 0$.

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