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Almost geodesic mappings of affinely connected spaces that preserve the Riemannian curvature*

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Abstract

In the present paper the authors give some conditions preserved Riemannian curvature tensor with respect to almost geodesic mappings of affinely connected spaces. It is noteworthy that these conditions are valid for other types of mappings. For the almost geodesic mappings of first type, when the Riemannian curvature tensor is invariant, the authors deduce a differential equations system of Cauchy type.

In addition the authors investigate almost geodesic mappings of first type, where the Weyl tensor of projective curvature is invariant and Riemannian tensor is not invariant.

Keywords: almost geodesic mapping, Riemannian curvature tensor

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1. Introduction

Several works [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] have been devoted to study almost geodesic mappings. These mappings are generalization of geodesic and quasigeodesic mappings, see [11, 12, 13, 17].

The basic concepts of almost geodesic curve and almost geodesic mapping of affinely connected spaces are introduced in paper [15] and included in the monographs [17, p. 156], [12, p. 457] and surveys [18, 4, 8, 10].

Definition 1.1. A curve $x(t)$ in an affinely connected space A_n is called an *almost geodesic curve* if there exists a plane $\tau(t)$ in every tangent space of the curve $x(t)$ such that:

(1) $\tau(t)$ are parallel translated along $x(t)$, and

(2) the tangent vector $\dot{x}(t)$ of the curve lies in $\tau(t)$.

Definition 1.2. A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called *almost geodesic mapping*, if under f any geodesic curve of A_n coincides with an almost geodesic curve of $\bar{A}_n$.

Theorem 1.3. *Diffeomorphism $f\colon A_n \to \bar{A}_n$ is almost geodesic mapping if and only if the deformation tensor of the affine connections $P^h_{ij}(x) \equiv \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ satisfies for any vector λ^h the following conditions:*

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\,P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\,\lambda^h$$

where

$$A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij}P^h_{k\alpha}, \tag{1.1}$$

Γ^h_{ij} $(\bar{\Gamma}^h_{ij})$ *are objects of affine connections of spaces A_n $(\bar{A}_n)$ respectively, a and b are some functions depend on x^h and λ^h and $x = (x^1, x^2, \dots, x^n)$ is a common system of coordinates. The symbol "," means covariant derivation with respect to A_n.*

Three types of almost geodesic mapping was discovered by Sinyukov [15, 16, 17, 18], he called them π_1, π_2 and π_3. In [2] it was proved that another almost geodesic mapping, if $n > 5$ does not exist.

Almost geodesic mapping π_1 is characterized by the following conditions:

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)},$$

where a_{ij} is a symmetric tensor, b_i is a vector, and the symbol (ijk) means symmetrization without division for the indices i, j, k.

2. Mappings of affinelly connected spaces that preserve the Riemann curvature tensor

If we give a diffeomorphism $f\colon A_n \to \bar{A}_n$, then the relation between Riemann curvature tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of A_n and $\bar{A}_n$ is the following [18, p. 78], [11, p. 86], [12, p. 184]:

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^\alpha_{i[k} P^h_{j]\alpha}, \tag{2.1}$$

where the symbol $[kj]$ denotes the alternalization for the indices k and j.

Using of (1.1) and (2.1) we have

Theorem 2.1. *A mapping preserves the Riemann curvature tensor if and only if it satisfies the condition*

$$A^h_{ijk} = A^h_{ikj}, \tag{2.2}$$

that is, the tensor A^h_{ijk} is to be symmetric in the indices j and k.

If the Riemann curvature tensor is preserved by the mapping, then, of course, Ricci tensor $R_{ij} = R^\alpha_{i\alpha j}$ and Weyl tensor of projective curvature

$$W^h_{ijk} = R^h_{ikj} - \frac{1}{n+1}\,\delta^h_i R_{[jk]} + \frac{1}{n^2-1}\,[(nR_{ij} + R_{ji})\delta^h_k - (nR_{ik} + R_{ki})\delta^h_j] \tag{2.3}$$

also are invariant under this mapping.

The condition of Theorem 1.3 is sufficient condition for preserving the Ricci tensor and Weyl tensor of projective curvature, but it is not necessary. Further on we give an example.

3. Special almost geodesic mappings of first type which preserve Riemannian tensor

Let be a mapping given between affinely connected spaces A_n and $\bar{A}_n$, which satisfies the condition:

$$P^h_{ij,k} + P^h_{ik,j} = -P^\alpha_{ij} P^h_{\alpha k} - P^\alpha_{ik} P^h_{\alpha j} + \delta^h_{(i} a_{jk)} \tag{3.1}$$

This mapping is a special case of almost geodesic mapping of first type.

Alternating equation (3.1) in i and j, we get

$$P^h_{ik,j} - P^h_{jk,i} = -P^\alpha_{ik} P^h_{\alpha j} + P^\alpha_{jk} P^h_{\alpha i}. \tag{3.2}$$

At now in (3.2), we exchange the indices i and k, we obtain

$$P^h_{ik,j} - P^h_{ji,k} = -P^\alpha_{ik} P^h_{\alpha j} + P^\alpha_{ji} P^h_{\alpha k}. \tag{3.3}$$

If we subtract equation (3.3) from equation (3.1), we have

$$2P^h_{ij,k} = -2P^\alpha_{ij} P^h_{\alpha k} + \delta^h_{(i} a_{jk)},$$

that is,

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = \delta^h_{(i} \tilde{a}_{jk)}, \tag{3.4}$$

where $\tilde{a}_{ij} = \frac{1}{2} a_{ij}$.

In this case the tensor $A^h_{ijk} = \delta^h_{(i} \tilde{a}_{jk)}$ is symmetric in indices j and k.

Using of Theorem 1.3 we have

Theorem 3.1. *The almost geodesic mapping (determined by* (3.1)*) preserves the Riemann curvature tensor* R^h_{ijk}.

If Riemann curvature tensor vanishes in an affine space, then we have the following

Theorem 3.2. *If an affine space* A_n *admits an almost geodesic mapping (determined by* (3.1)*) into* $\bar{A}_n$*, then* $\bar{A}_n$ *is also an affine space.*

So affinely spaces are closed under almost geodesic mapping (determined by (3.1)).

From equation (3.1) we obtained the equation (3.4). Equation (3.4) is a system of Cauchy type for deformation tensor. We can find it's integrability conditions.

We differentiate covariantly equation (3.4) by x^m, further on, we change the indices k and m, using of Ricci identities we have

$$\delta^h_i \tilde{a}_{j[k,m]} + \delta^h_j \tilde{a}_{i[k,m]} + \delta^h_{[k} \tilde{a}_{ij,lm]} = P^\alpha_{ij} R^h_{\alpha km} + P^h_{\alpha(j} R^\alpha_{i)km} + a_{\tilde{j}[m} P^h_{k]i} + \tilde{a}_{i[m} P^h_{k]j}. \tag{3.5}$$

After transvecting of integrability conditions (3.5) by indices h and m we obtain

$$\begin{aligned} \tilde{a}_{jk,i} + \tilde{a}_{ik,j} - (n+1)\tilde{a}_{ij,k} = -P^\alpha_{ij} R_{\alpha k} + P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha i} R^\alpha_{jk\beta} \\ + \tilde{a}_{j\alpha} P^\alpha_{ki} - \tilde{a}_{jk} P^\alpha_{\alpha i} + \tilde{a}_{i\alpha} P^\alpha_{jk} - \tilde{a}_{ik} P^\alpha_{j\alpha}. \end{aligned} \tag{3.6}$$

Alternating equation (3.6) in k and j, we obtain

$$\begin{aligned} \tilde{a}_{ij,k} = \tilde{a}_{ik,j} + \tfrac{1}{n+2}(-P^\alpha_{ij} R_{\alpha k} + P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} - \\ P^\beta_{\alpha i} R^\alpha_{jk\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - \tilde{a}_{j\alpha} P^\alpha_{ki} + \tilde{a}_{k\alpha} P^\alpha_{ij} + \tilde{a}_{ik} P^\alpha_{j\alpha} - \tilde{a}_{ij} P^\alpha_{k\alpha}). \end{aligned} \tag{3.7}$$

In equation we exchange the indices k and i, we get

$$\begin{aligned} \tilde{a}_{kj,i} = \tilde{a}_{ik,j} + \tfrac{1}{n+2}(-P^\alpha_{kj} R_{\alpha i} + P^\alpha_{ki} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ki\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - \\ P^\beta_{\alpha k} R^\alpha_{ji\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} - \tilde{a}_{j\alpha} P^\alpha_{ik} + \tilde{a}_{i\alpha} P^\alpha_{kj} + \tilde{a}_{ki} P^\alpha_{j\alpha} - \tilde{a}_{kj} P^\alpha_{i\alpha}). \end{aligned} \tag{3.8}$$

Substituting equation (3.7) and (3.8) into (3.6), we have

$$\begin{aligned} \tilde{a}_{ik,j} = \tfrac{1}{(n-1)(n+2)}(-n(P^\alpha_{ik} R_{\alpha j} + P^\beta_{\alpha(k} R^\alpha_{i)j\beta})) - R_{\alpha(k} P^\alpha_{i)j} - P^\beta_{\alpha j} R^\alpha_{(ik)\beta} - \\ P^\beta_{\alpha(i} R^\alpha_{|j|k)\beta} + (n+1)(\tilde{a}_{j(i} P^\alpha_{k)\alpha} - \tilde{a}_{\alpha(i} P^\alpha_{k)j}) + 2(\tilde{a}_{ik} P^\alpha_{j\alpha} - \tilde{a}_{j\alpha} P^\alpha_{ik})). \end{aligned} \tag{3.9}$$

Equation (3.4) and (3.9) are a system of Cauchy type for the function $P^h_{ij}(x)$ and $\tilde{a}_{ij}(x)$, which satisfy the following

$$P^h_{ij}(x) = P^h_{ji}(x), \quad \tilde{a}_{ij}(x) = \tilde{a}_{ji}(x). \tag{3.10}$$

So is proved the

Theorem 3.3. *An affinely connected space A_n admits an almost geodesic mapping (determined by* (3.1)*) into an affinely connected space $\bar{A}_n$ if and only if in A_n exist solution functions $P^h_{ij}(x)$ and $\tilde{a}_{ij}(x)$ for equation system of Cauchy type* (3.4), (3.9) *and* (3.10).

4. Special almost geodesic mappings of first type which preserve Weyl tensor of projective curvature but does not preserve Riemann curvature tensor

Let be the $f\colon \bar{A}_n \to \bar{A}_n$ mapping given, which satisfies the following condition

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = \delta^h_k a_{ij}, \tag{4.1}$$

where a_{ij} is a symmetric tensor.

It is well known, that this above mentioned mapping is an almost geodesic mapping of first type.

The tensor A^h_{ijk} on the basis of (4.1) is equal to $\delta^h_k a_{ij}$. If the tensor $a_{ij} \not\equiv 0$, then the tensor A^h_{ijk} is not symmetric in indices j and k. So, in general the mapping (determined by (4.1)) does not preserve the Riemannian curvature tensor.

Using of (2.1) and (4.1) we get

$$\bar{R}^h_{ijk} = R^h_{ijk} - a_{i[j}\delta^h_{k]}. \tag{4.2}$$

It is easy to see, that after transvecting (4.2) in indices h and k, we have

$$a_{ij} = \frac{1}{n-1}\,(\bar{R}_{ij} - R_{ij}).$$

From the last formulae, symmetry a_{ij}, (2.3) and (4.2) we get

$$\bar{W}_{ij} = W_{ij}, \quad \bar{\tilde{W}}{}^h_{ijk} = \tilde{W}^h_{ijk}, \quad \text{and} \quad \bar{W}^h_{ijk} = W^h_{ijk}$$

where

$$W_{ij} = R_{ij} - R_{ji} \quad \text{and} \quad \tilde{W}^h_{ijk} = R^h_{ijk} + \frac{1}{n-1} R_{i[j}\delta^h_{k]}.$$

The W_{ij} and $\bar{W}_{ij}$ are tensors of type $\binom{0}{2}$ in the space A_n and $\bar{A}_n$ respectively. The $\tilde{W}^h_{ijk}$ and $\bar{\tilde{W}}{}^h_{ijk}$ are tensors of type $\binom{1}{3}$ in the space A_n and $\bar{A}_n$ respectively. The W^h_{ijk} and $\bar{W}^h_{ijk}$ are Weyl tensors of projective curvature of A_n and $\bar{A}_n$ respectively.

Finally we obtain

Theorem 4.1. *Tensors W_{ij}, $\tilde{W}^h_{ijk}$ and W^h_{ijk} are invariants under almost geodesic mapping* (*determined by* (4.1)).

From Theorem 3.3 follows

Theorem 4.2. *If a projective-euclidean or equiaffinely space A_n admits almost geodesic mapping* (*determined by* (4.1)) *into an affinely connected space $\bar{A}_n$, then $\bar{A}_n$ is a projective-euclidean or equiaffinely space respectively.*

The proof of Theorem 4.1 follows from facts, that the Weyl tensor vanishes in projective-euclidean space, and the tensor is equal to zero in equiaffinely space.

So, using of Theorem 4.1, we obtain, that the projective-euclidean and equiaffinely spaces are closed sets under almost geodesic mapping (determined by (4.1)).

For almost geodesic mapping of first type, which determined by (4.1), the tensor A^h_{ijk} is equal to $\delta^h_i a_{ij}$. If $a_{ij} \not\equiv 0$, then the A^h_{ijk} tensor is not symmetric in indices j and k.

So, the mapping (determined by (4.1)) does not preserve the Riemann curvature tensor, but the Weyl tensor is invariant object.

Consider the equations (4.1) as a system of Cauchy-type for unknown P^h_{ij}, find it's integrability condition. At first, we differentiate covariantly equation (4.1) in x^m, further on we alternate it in indices k and m. After transvecting integrability condition of equation (4.1) in indices h and m, we obtain

$$(n-1)a_{ij,k} = P^\alpha_{ij} R_{\alpha k} - P^\beta_{\alpha(i} R^\alpha_{j)\beta k} - (n-1) P^\alpha_{ij} a_{\alpha k} \tag{4.3}$$

So equation (4.1) and (4.3) in a space A_n give a system of Cauchy type for unknown functions $P^h_{ij}(x)$ and $a_{ij}(x)$, which satisfies algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x) \tag{4.4}$$

Therefore

Theorem 4.3. *An affinely connected space admits almost geodesic mapping of first type (determined by* (4.1)*) into an affinely connected space $\bar{A}_n$ if and only if in A_n there exist solution $P^h_{ij}(x)$ and $a_{ij}(x)$ for system of Cauchy type equations* (4.1), (4.3) *and* (4.4).

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