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Solutions of some particular pexiderized digital filtering functional equation

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Abstract

Consider the pexiderized digital filtering functional equation

$$f_1(x+t,y+t)+f_2(x-t,y)+f_3(x,y-t) = f_4(x-t,y-t)+f_5(x+t,y)+f_6(x,y+t).$$

We determine three kinds of solutions, namely, biadditive, symmetric and skew-symmetric solution functions, subject to different sets of conditions on the functions involved.

Keywords: digital filtering functional equation, pexiderized form, biadditive function, symmetric function, skew-symmetric function.

MSC: 39B52, 39A14

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1. Introduction

Throughout let G be an abelian group which is 2-solvable, i.e., the equation $2u = v$ is solvable. A function $A\colon G \times G \to \mathbb{C}$ is said to be

- symmetric if $A(x, y) = A(y, x)$;
- skew-symmetric if $A(x, y) = -A(y, x)$;
- additive if $A(x + y) = A(x) + A(y)$:
- biadditive if it is additive in each of its variables.

Some well-known facts that we shall use implicitly are

- symmetric, skew-symmetric and biadditive properties are preserved under addition;
- if $A(x, y)$ is skew-symmetric, then $A(x, x) = 0$;
- if $A(x, y)$ is biadditive, then $A(x, 0) = 0 = A(0, y)$, $A(-x, y) = -A(x, y) = A(x, -y)$ and $A\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ is skew-symmetric.

In [4] the following functional equation related to digital filtering (see Proposition 1.2 below) is solved

$$f(x+t, y+t) + f(x-t, y) + f(x, y-t) = f(x-t, y-t) + f(x, y+t) + f(x+t, y),$$

where $f\colon G \times G \to \mathbb{C}$ and $x, y, t \in G$. Here we consider its pexiderized version, which is

$$\begin{aligned} &f_1(x+t, y+t) + f_2(x-t, y) + f_3(x, y-t) \\ &\qquad = f_4(x-t, y-t) + f_5(x+t, y) + f_6(x, y+t) \quad (x, y, t \in G), \end{aligned} \tag{PDF}$$

where $f_1, f_2, f_3, f_4, f_5, f_6\colon G \times G \to \mathbb{C}$. Since solving (PDF) generally seems quite difficult, we are content here to exhibit three kinds of solution functions of (PDF), namely, biadditive, symmetric and skew-symmetric solution functions. The case of biadditive functions is most satisfactory as complete shapes of solutions can be determined, while the remaining two cases are harder and we are forced to impose some more restrictions, which arise from certain symmetry of the functions involved. subject to three different sets of conditions on the functions involved. We will also appeal to the following results in [3], [4] and [1].

Proposition 1.1. *[3] If $f\colon G \to \mathbb{C}$ satisfies*

$$f(x+t, y+t) = f(x+t, y) + f(x, y+t) - f(x, y) \quad (x, y, t \in G),$$

then

$$f(x, y) = \phi(x) + \psi(y) + A(x, y)$$

for arbitrary mappings $\phi, \psi\colon G \to \mathbb{C}$ and arbitrary skew-symmetric biadditive map $A\colon G \times G \to \mathbb{C}$.

Proposition 1.2. *[4] The function $f\colon G\times G\to\mathbb{C}$ satisfies the functional equation*

$$f(x+t,y+t)+f(x-t,y)+f(x,y-t)=f(x-t,y-t)+f(x,y+t)+f(x+t,y)$$

for all x,y,t in G if and only if

$$f(x,y)=B(x,y)+\phi(x)+\psi(y)+\chi(x-y)$$

holds for all x,y in G, where $B\colon G\times G\to\mathbb{C}$ is biadditive and $\phi,\psi,\chi\colon G\to\mathbb{C}$ are arbitrary functions.

Proposition 1.3. *[1] If $f_1,f_2,f_3,f_4\colon G\times G\to\mathbb{C}$ satisfy the functional equation*

$$f_1(x+t,y+s)+f_2(x-t,y-s)=f_3(x+s,y-t)+f_4(x-s,y+t)\quad(x,y,s,t\in G),$$

then f_1,f_2,f_3 and f_4 are given by

$$f_1=w+h,\ \ f_2=w-h,\ \ f_3=w+k,\ \ f_4=w-k$$

where $w\colon G\times G\to\mathbb{C}$ is an arbitrary solution of the functional equation

$$w(x+t,y+s)+w(x-t,y-s)=w(x+s,y-t)+w(x-s,y+t)\quad(x,y,s,t\in G)$$

and $h,k\colon G\times G\to\mathbb{C}$ are arbitrary solutions of the system of difference functional equations

$$\Delta^3_{y,t}g(x,y)=0,\quad \Delta^3_{x,t}g(x,y)=0\quad(x,y,t\in G),$$

where the two partial difference operators are defined by

$$\Delta_{x,t}g(x,y)=g(x+t,y)-g(x,y),\ \ \Delta_{y,t}g(x,y)=g(x,y+t)-g(x,y).$$

2. Auxiliary results

It is convenient to introduce translation operators X^t and Y^t for $t\in G$, which are defined by

$$X^tf(x,y)=f(x+t,y),\quad Y^tf(x,y)=f(x,y+t).$$

In particular, $X^0=Y^0=1$ denote the identity operator.

Lemma 2.1. *Let $f_1,f_2,f_3,f_4,f_5,f_6\colon G\times G\to\mathbb{C}$, and for $i,j\in\{1,\dots,6\}$ with $i\neq j$, put*

$$S^{(i,j)}(x,y)=\frac{1}{2}\left\{f_i(x,y)+f_j(x,y)\right\},\ \ D^{(i,j)}(x,y)=\frac{1}{2}\left\{f_i(x,y)-f_j(x,y)\right\}.$$

Assume that f_1,f_2,f_3,f_4,f_5,f_6 satisfy (PDF).
A) Then

$$X^tY^tf_1+X^{-t}f_2+Y^{-t}f_3=X^{-t}Y^{-t}f_4+X^tf_5+Y^tf_6\tag{2.1}$$

$$\left(X^tY^t - X^{-t}Y^{-t}\right)S^{(1,4)} + \left(X^{-t} - X^t\right)S^{(2,5)} + \left(Y^{-t} - Y^t\right)S^{(3,6)} = 0 \quad (2.2)$$

$$\left(X^tY^t + X^{-t}Y^{-t}\right)D^{(1,4)} + \left(X^{-t} + X^t\right)D^{(2,5)} + \left(Y^{-t} + Y^t\right)D^{(3,6)} = 0. \quad (2.3)$$

B) If, in addition, $f_1, f_2, f_3, f_4, f_5, f_6$ *are symmetric or skew-symmetric functions, then*

$$(X^{-t} - Y^{-t})D^{(2,3)} = \left(X^t - Y^t\right)D^{(5,6)} \quad (2.4)$$

$$X^tY^t(2f_1) + (X^{-t} + Y^{-t})S^{(2,3)} = X^{-t}Y^{-t}(2f_4) + (X^t + Y^t)S^{(5,6)} \quad (2.5)$$

$$X^tY^tS^{(1,4)} + X^{-t}S^{(2,6)} + Y^{-t}S^{(3,5)} = X^{-t}Y^{-t}S^{(1,4)} + X^tS^{(3,5)} + Y^tS^{(2,6)}. \quad (2.6)$$

Proof. A) Writing (PDF) in the operator form, we get

$$\begin{aligned} &X^tY^tf_1(x,y) + X^{-t}f_2(x,y) + Y^{-t}f_3(x,y) \\ &= X^{-t}Y^{-t}f_4(x,y) + X^tf_5(x,y) + Y^tf_6(x,y), \end{aligned}$$

which is (2.1). Replacing t by $-t$, we get

$$\begin{aligned} &X^{-t}Y^{-t}f_1(x,y) + X^tf_2(x,y) + Y^tf_3(x,y) \\ &= X^tY^tf_4(x,y) + X^{-t}f_5(x,y) + Y^{-t}f_6(x,y). \end{aligned}$$

The relations (2.2) and (2.3) follow from subtracting, respectively adding, the last two equations and rearranging terms.
B) Using the fact that $f_1, f_2, f_3, f_4, f_5, f_6$ are symmetric or skew-symmetric, (2.1) becomes

$$X^tY^tf_1 + Y^{-t}f_2 + X^{-t}f_3 = X^{-t}Y^{-t}f_4 + Y^tf_5 + X^tf_6. \quad (2.7)$$

Replacing t by $-t$, we get

$$X^tY^tf_4 + Y^{-t}f_5 + X^{-t}f_6 = X^{-t}Y^{-t}f_1 + Y^tf_2 + X^tf_3. \quad (2.8)$$

The relations (2.4) and (2.5) follow from subtracting, respectively, adding (2.7) and (2.1). The relation (2.6) comes from adding (2.8) and (2.1). □

3. Bi-additive solutions

In this section, biadditive solutions of (PDF) are completely determined.

Theorem 3.1. *If* $f_1, f_2, f_3, f_4, f_5, f_6$ *are biadditive functions satisfying* (PDF), *then*

$$\begin{aligned} f_1(x,y) &= B(x,y) - C(x,y) - D(x,y), \\ f_2(x,y) &= B(x,y) + C(x,y), \\ f_3(x,y) &= B(x,y) + D(x,y), \end{aligned}$$

$$f_4(x,y) = B(x,y) + C(x,y) + D(x,y),$$
$$f_5(x,y) = B(x,y) - C(x,y),$$
$$f_6(x,y) = B(x,y) - D(x,y),$$

where $B, C, D\colon G \times G \to \mathbb{C}$ *are arbitrary biadditive functions satisfying*

$$C(t,t) + D(t,t) = 0 \quad (t \in G).$$

Proof. Rewriting (2.2), we get

$$S^{(1,4)}(x+t,y+t) - S^{(1,4)}(x-t,y-t) + S^{(2,5)}(x-t,y) - S^{(2,5)}(x+t,y) + S^{(3,6)}(x,y-t) - S^{(3,6)}(x,y+t) = 0.$$

Since $S^{(1,4)}, S^{(2,5)}, S^{(3,6)}$ are biadditive, simplifying we get

$$S^{(1,4)}(t,y) - S^{(2,5)}(t,y) = -S^{(1,4)}(x,t) + S^{(3,6)}(x,t). \tag{3.1}$$

Replacing t by x, we have

$$S^{(1,4)}(x,y) - S^{(2,5)}(x,y) = -S^{(1,4)}(x,x) + S^{(3,6)}(x,x) =: \mathcal{S}_1(x).$$

Substituting $y = 0$, we get $\mathcal{S}_1(x) = 0$, and so

$$S^{(1,4)}(x,y) = S^{(2,5)}(x,y). \tag{3.2}$$

Similarly, replacing t by y in (3.1) and substituting $x = 0$, we get

$$S^{(1,4)}(x,y) = S^{(3,6)}(x,y). \tag{3.3}$$

From (3.2) and (3.3), we set

$$S^{(2,5)}(x,y) = S^{(1,4)}(x,y) = S^{(3,6)}(x,y) =: B(x,y). \tag{3.4}$$

On the other hand, rewriting (2.3), we get

$$D^{(1,4)}(x+t,y+t) + D^{(1,4)}(x-t,y-t) + D^{(2,5)}(x-t,y) + D^{(2,5)}(x+t,y) + D^{(3,6)}(x,y-t) + D^{(3,6)}(x,y+t) = 0.$$

Since $D^{(1,4)}, D^{(2,5)}, D^{(3,6)}$ are biadditive, simplifying we get

$$D^{(1,4)}(x,y) + D^{(1,4)}(t,t) + D^{(2,5)}(x,y) + D^{(3,6)}(x,y) = 0.$$

Setting $y = 0$, we have

$$D^{(1,4)}(t,t) = 0, \tag{3.5}$$

and so

$$D^{(1,4)}(x,y) + D^{(2,5)}(x,y) + D^{(3,6)}(x,y) = 0.$$

Putting

$$D^{(2,5)}(x,y) =: C(x,y) \tag{3.6}$$

$$D^{(3,6)}(x,y) =: D(x,y) \tag{3.7}$$

we arrive at

$$D^{(1,4)}(x,y) = -C(x,y) - D(x,y). \tag{3.8}$$

From (3.5), we have $C(t,t) + D(t,t) = 0$. The desired result follows from solving (3.4),(3.6),(3.7) and (3.8) for $f_1, \dots, f_6$. □

4. Symmetric solutions

In this section, we consider solutions of (PDF) which are symmetric functions. This case is much more complicated than the previous one and so we start with several lemmas.

Lemma 4.1. *Let $B\colon G \times G \to \mathbb{C}$ be biadditive and let $\phi, \psi, \chi\colon G \to \mathbb{C}$ be arbitrary functions. If*

$$f(x,y) = B(x,y) + \phi(x) + \psi(y) + \chi(x-y) \tag{4.1}$$

is symmetric, then

$$f(x,y) = B(x,y) + \{\phi + \psi(x) + \psi(y)\} + \{\chi(-x) - \chi(x) + \chi(x-y)\},$$

where the biadditive function $B(x,y)$ is symmetric, ϕ is a complex constant, and $\chi(-x) - \chi(x)$ is an additive function of x.

Proof. Since $f(x,y)$ is symmetric, equating $f(x,y) = f(y,x)$, we get

$$B(x,y) + \phi(x) + \psi(y) + \chi(x-y) = B(y,x) + \phi(y) + \psi(x) + \chi(y-x). \tag{4.2}$$

Substituting $y = 0$, using $B(x,0) = 0 = B(0,x)$, putting $\phi = \phi(0) - \psi(0)$ and simplifying, we have

$$\phi(x) = \phi + \psi(x) + \chi(-x) - \chi(x).$$

Replacing this $\phi(x)$ in (4.2) and simplifying we get

$$B(x,y) + \chi(-x) - \chi(x) + \chi(x-y) = B(y,x) + \chi(-y) - \chi(y) + \chi(y-x). \tag{4.3}$$

Substituting $y = x - z$ and using biadditivity, we get

$$B(z,x) + \chi(-x) - \chi(x) + \chi(z) = B(x,z) + \chi(z-x) - \chi(x-z) + \chi(-z).$$

Replacing z by y, we get

$$B(y,x) + \chi(-x) - \chi(x) + \chi(y) = B(x,y) + \chi(y-x) - \chi(x-y) + \chi(-y). \tag{4.4}$$

Subtracting (4.4) from (4.3), we deduce that $B(x, y) = B(y, x)$, i.e., B is symmetric. Adding (4.3) and (4.4), and rearranging we deduce that

$$\{\chi(-x) - \chi(x)\} + \{\chi(x - y) - \chi(y - x)\} = \chi(-y) - \chi(y),$$

i.e., $\chi(-x) - \chi(x)$ is an additive function of x. Incorporating all the information obtained, the result follows. □

Lemma 4.2. *Let the notation be as in Lemma 2.1. If $f_1, f_2, f_3, f_4, f_5, f_6$ are symmetric (or skew-symmetric) functions satisfying* (PDF), *then*

$$-\frac{1}{2}\{f_2(x, y) - f_3(x, y)\} =: -D^{(2,3)}(x, y) = w(x, y) + k(x, y) \tag{4.5}$$

$$\frac{1}{2}\{f_5(x, y) - f_6(x, y)\} =: D^{(5,6)}(x, y) = w(x, y) - k(x, y). \tag{4.6}$$

where the functions w and k are as described in Proposition 1.3.

Proof. Rewriting (2.4), we get

$$D^{(5,6)}(x + t, y) - D^{(2,3)}(x - t, y) = D^{(5,6)}(x, y + t) - D^{(2,3)}(x, y - t).$$

Using Proposition 1.3 with $s = 0,\ f_1 = D^{(5,6)} = f_4,\ f_2 = -D^{(2,3)} = f_3$, we have

$$w + h = f_1 = D^{(5,6)} = f_4 = w - k,\ w - h = f_2 = -D^{(2,3)} = f_3 = w + k,$$

i.e., $h = -k$, and the result follows. □

Lemma 4.3. *Let $f_1, f_2, f_3, f_4, f_5, f_6 \colon G \times G \to \mathbb{C}$, and let*

$$K(x, y) := f_2 + f_5 - f_3 - f_6, \quad H(x, y) := f_2 - f_5 - f_3 + f_6.$$

If $f_1, f_2, f_3, f_4, f_5, f_6$ are symmetric functions satisfying (PDF), *then*

$$K(x, y) = \alpha_K(x + y) + \beta \tag{4.7}$$

$$H(x, y) = \alpha_H(x + y) + \frac{1}{2}\{\beta_H(x - y) + \beta_H(y - x)\}, \tag{4.8}$$

where $\alpha_K, \alpha_H, \beta_H \colon G \to \mathbb{C}$ are arbitrary functions, and β is a complex constant.

Proof. Using symmetry in (2.2), we get

$$\left(X^t Y^t - X^{-t} Y^{-t}\right) S^{(1,4)} + \left(Y^{-t} - Y^t\right) S^{(2,5)} + \left(X^{-t} - X^t\right) S^{(3,6)} = 0. \tag{4.9}$$

Subtracting (4.9) from (2.2) and rearranging, we get

$$\left(X^{-t} - X^t\right) K(x, y) = \left(Y^{-t} - Y^t\right) K(x, y). \tag{4.10}$$

Operating both sides of (4.10) by $X^{-t} - X^t$ and using (4.10) again, we get

$$\left(X^{-2t} - 2 + X^{2t}\right) K(x, y) = \left(Y^{-t} - Y^t\right)\left(X^{-t} - X^t\right) K(x, y)$$

$$= \left(Y^{-2t} - 2 + Y^{2t}\right) K(x,y).$$

Simplifying and replacing $2t$ by t, we have

$$\left(X^{-t} + X^{t}\right) K(x,y) = \left(Y^{-t} + Y^{t}\right) K(x,y). \tag{4.11}$$

Defining the function $K_1 \colon G \times G \to \mathbb{C}$ by

$$K_1(x,y) := K\left(\frac{x+y}{2}, \frac{x-y}{2}\right), \text{ or equivalently, } \quad K(x,y) = K_1(x+y, x-y)$$

and rewriting (4.11) in terms of K_1, we get

$$\begin{aligned} &K_1(x+y-t, x-y-t) + K_1(x+y+t, x-y+t) \\ &= K_1(x+y-t, x-y+t) + K_1(x+y+t, x-y-t). \end{aligned}$$

Putting $u = x+y-t,\ v = x-y-t$, this last equation becomes

$$K_1(u,v) + K_1(u+2t, v+2t) = K_1(u, v+2t) + K_1(u+2t, v).$$

Replacing $2t$ by t and rearranging, we get

$$K_1(u+t, v+t) = K_1(u+t, v) + K_1(u, v+t) - K_1(u,v),$$

which is the McKiernan's functional equation mentioned in Proposition 1.1, whose solution is of the form

$$K_1(x,y) = \alpha_K(x) + \beta_K(y) + A_K(x,y),$$

with the functions α_K, β_K, A_K as stated above. Reverting back to K, we get

$$K(x,y) = \alpha_K(x+y) + \beta_K(x-y) + A_K(x+y, x-y).$$

Since A_K is skew-symmetric and biadditive, the shape of K reduces to

$$K(x,y) = \alpha_K(x+y) + \beta_K(x-y) - 2A_K(x,y). \tag{4.12}$$

Since K is symmetric, equating $K(x,y) = K(y,x)$, we get

$$\beta_K(x-y) - \beta_K(y-x) = 4A_K(x,y). \tag{4.13}$$

Substituting $y = 0$, we get $\beta_K(x) - \beta_K(-x) = 0$, i.e., β_K is an even function. From (4.10) and (4.11), we see at once that $X^t K(x,y) = Y^t K(x,y)$. Using this and (4.12), we get

$$\begin{aligned} &\alpha_K(x+y+t) + \beta_K(x-y+t) - 2A_K(x+t, y) \\ &= \alpha_K(x+y+t) + \beta_K(x-y-t) - 2A_K(x, y+t). \end{aligned}$$

Since A_K is biadditive and skew-symmetric, this last relation simplifies to

$$\beta_K(x-y+t)-\beta_K(x-y-t)=2A_K(t,x+y).$$

Putting $x=y$, and using the fact that β_K is an even function, we have $A_K(x,y)=0$, and so $\beta_K(x-y+t)-\beta_K(x-y-t)=0$. Taking $x-y-t=0$, we deduce that $\beta_K(x,y)=\beta$, a constant. The shape of K follows by collecting all the information found.

The determination of H proceeds analogously. Using symmetry in (2.3), we get

$$\left(X^tY^t+X^{-t}Y^{-t}\right)D^{(1,4)}+\left(Y^{-t}+Y^t\right)D^{(2,5)}+\left(X^{-t}+X^t\right)D^{(3,6)}=0. \quad (4.14)$$

Subtracting (4.14) from (2.3) and rearranging, we get

$$\left(X^{-t}+X^t\right)H(x,y)=\left(Y^{-t}+Y^t\right)H(x,y). \quad (4.15)$$

Defining the function $H_1\colon G\times G\to\mathbb{C}$ by

$$H_1(x,y):=H\left(\frac{x+y}{2},\frac{x-y}{2}\right),\ \text{ or equivalently,}\quad H(x,y)=H_1(x+y,x-y)$$

and rewriting (4.15) in terms of H_1, we get

$$\begin{aligned}&H_1(x+y-t,x-y-t)+H_1(x+y+t,x-y+t)\\&=H_1(x+y-t,x-y+t)+H_1(x+y+t,x-y-t).\end{aligned}$$

Putting $u=x+y-t,\ v=x-y-t$, then replacing $2t$ by t and rearranging, we get

$$H_1(u+t,v+t)=H_1(u+t,v)+H_1(u,v+t)-H_1(u,v),$$

which is the McKiernan's functional equation mentioned in Proposition 1.1, whose solution is of the form

$$H_1(x,y)=\alpha_H(x)+\beta_H(y)+A_H(x,y),$$

with the functions α_H,β_H,A_H as stated above. Reverting back to H, we get

$$H(x,y)=\alpha_H(x+y)+\beta_H(x-y)+A_H(x+y,x-y).$$

As in the previous case, using the fact that A_H is skew-symmetric and biadditive, the shape of H reduces to

$$H(x,y)=\alpha_H(x+y)+\beta_H(x-y)-2A_H(x,y).$$

Since H is symmetric, equating $H(x,y)=H(y,x)$, we get $\beta_H(x-y)-\beta_H(y-x)=4A_H(x,y)$. Incorporating all these details, the shape of H follows. □

Remark 4.1. R1) Setting $t=0$ in (4.14), and simplifying, we get

$$(f_1-f_4)+(f_2-f_5)+(f_3-f_6)=0.$$

R2) From Lemma 4.3 and Lemma 4.2, we have

$$\begin{aligned}&\alpha_K(x+y)+\beta\\&=K=f_2+f_5-f_3-f_6=-4k(x,y) \qquad (4.16)\\&\alpha_H(x+y)+\frac{1}{2}\{\beta_H(x-y)+\beta_H(y-x)\}\\&=H=f_2-f_5-f_3+f_6=-4w(x,y). \qquad (4.17)\end{aligned}$$

Throughout the rest of this section, we shall deal only with the case where $f_1=f_4$, which, by remark R1), gives rise to

$$f_2-f_5+f_3-f_6=0.$$

Combining with (4.16) and (4.17), we get

$$f_2-f_6=-2k,\ \ f_3-f_6=2w \qquad (4.18)$$

Theorem 4.4. *Assume* $f_1,f_2,f_3,f_4,f_5,f_6\colon G\times G\to\mathbb{C}$ *are symmetric functions satisfying* (PDF).
I. If

$$f_1=f_4=\frac{1}{2}(f_2+f_3),$$

then there are biadditive, symmetric function $B(x,y)\colon G\times G\to\mathbb{C}$*, two constants* ϕ,β_1*, arbitrary functions* $\psi,\alpha_1,\alpha_2,\beta_2$ *and* $\chi\colon G\to\mathbb{C}$ *with* $\chi(x)-\chi(-x)$ *being an additive function in* x *such that*

$$\begin{aligned}f_1(x,y)&=f_4(x,y)=B(x,y)+\{\psi(x)+\psi(y)+\phi\}+\{\chi(-x)-\chi(x)+\chi(x-y)\}\\f_2(x,y)&=f_1(x,y)+\{\alpha_1(x+y)+\beta_1\}+\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\\f_3(x,y)&=f_1(x,y)-\{\alpha_1(x+y)+\beta_1\}-\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\\f_5(x,y)&=f_1(x,y)+\{\alpha_1(x+y)+\beta_1\}-\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\\f_6(x,y)&=f_1(x,y)-\{\alpha_1(x+y)+\beta_1\}+\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}.\end{aligned}$$

II. If

$$f_1=f_4=\frac{1}{2}(f_2+f_6)=\frac{1}{2}(f_3+f_5),$$

then there are biadditive, symmetric function $B(x,y)\colon G\times G\to\mathbb{C}$*, two constants* ϕ,β_1*, arbitrary functions* ψ,α_1 *and* $\chi\colon G\to\mathbb{C}$ *with* $\chi(x)-\chi(-x)$ *being an additive function in* x *such that*

$$\begin{aligned}f_1(x,y)&=f_4(x,y)=B(x,y)+\{\psi(x)+\psi(y)+\phi\}+\{\chi(-x)-\chi(x)+\chi(x-y)\}\\f_2(x,y)&=f_5(x,y)=f_1(x,y)+\{\alpha_1(x+y)+\beta_1\}\end{aligned}$$

$$f_3(x,y)=f_6(x,y)=f_1(x,y)-\{\alpha_1(x+y)+\beta_1\}\,.$$

III. If

$$f_1=f_4=\frac{1}{2}\left(f_2+f_5\right)=\frac{1}{2}\left(f_3+f_6\right),$$

then there are biadditive, symmetric function $B(x,y)\colon G\times G\to\mathbb{C}$, a constant ϕ, arbitrary functions ψ,α_2,β_2 and $\chi\colon G\to\mathbb{C}$ with $\chi(x)-\chi(-x)$ being an additive function in x such that

$$\begin{aligned}
f_1(x,y)&=f_4(x,y)=B(x,y)+\{\psi(x)+\psi(y)+\phi\}+\{\chi(-x)-\chi(x)+\chi(x-y)\}\\
f_2(x,y)&=f_6(x,y)=f_1(x,y)+\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\\
f_3(x,y)&=f_5(x,y)=f_1(x,y)-\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\,.
\end{aligned}$$

IV. If

$$f_1=f_4=f_6\quad\text{and}\quad K(x,x)-H(x,x)=c$$

where c is a constant, then there are biadditive, symmetric function $B(x,y)\colon G\times G\to\mathbb{C}$, two constants ϕ,β_1, arbitrary functions $\psi,\alpha_1,\alpha_2,\beta_2$, $\chi\colon G\to\mathbb{C}$ with $\alpha_1(x)-\alpha_2(x)$ being a constant and $\chi(x)-\chi(-x)$ being an additive function in x such that

$$\begin{aligned}
f_1(x,y)&=f_4(x,y)=f_6(x,y)=B(x,y)+\{\psi(x)+\psi(y)+\phi\}\\
&\qquad\qquad+\{\chi(-x)-\chi(x)+\chi(x-y)\}\\
f_2(x,y)&=f_1(x,y)+\{\alpha_1(x+y)+\beta_1\}\\
f_3(x,y)&=f_1(x,y)-\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\\
f_5(x,y)&=f_1(x,y)+\{\alpha_1(x+y)+\beta_1\}-\{\alpha_2(x+y)+\beta_2(x-y)+\beta_2(y-x)\}\,.
\end{aligned}$$

Proof. I. Using $f_1=f_4,\ f_2+f_3=f_5+f_6$, substituting $g=2f_1=2f_4=f_2+f_3=f_5+f_6$ in (2.5) and simplifying, we get

$$\begin{aligned}
&g(x+t,y+t)+g(x-t,y)+g(x,y-t)\\
&=g(x-t,y-t)+g(x,y+t)+g(x+t,y),
\end{aligned}$$

which is the Sahoo-Székelyhidi's functional equation mentioned in Proposition 1.2, and its solution is of the form

$$g(x,y)=B_1(x,y)+\phi_1(x)+\psi_1(y)+\chi_1(x-y),$$

where $B_1\colon G\times G\to\mathbb{C}$ is biadditive and $\phi_1,\psi_1,\chi_1\colon G\to\mathbb{C}$ are arbitrary functions. Since g is symmetric, applying Lemma 4.1, we deduce that

$$g(x,y)=B_1(x,y)+\{\phi_1+\psi_1(x)+\psi_1(y)\}+\{\chi_1(-x)-\chi_1(x)+\chi_1(x-y)\}\,,\quad(4.19)$$

where ϕ_1 is a constant and $\chi_1(-x)-\chi_1(x)$ is an additive function of x, and by putting

$$B=\frac{B_1}{2},\ \phi=\frac{\phi_1}{2},\ \psi=\frac{\psi_1}{2},\ \chi=\frac{\chi_1}{2},$$

this gives the shapes of f_1 and f_4. Using (4.5), (4.16) and (4.17), we get

$$\begin{aligned}2f_3 &= (f_2+f_3)+(-f_2+f_3) = g+2(w+k)\\ &= g(x,y)-\frac{1}{4}\left\{\alpha_H(x+y)+\frac{1}{4}\left(\beta_H(x-y)+\beta_H(y-x)\right)\right\}\\ &\qquad -\frac{1}{4}\left\{\alpha_K(x+y)+\beta\right\}\end{aligned}$$

and the shape of f_3 follows by putting $\alpha_1=\frac{1}{4}\alpha_K,\ \beta_1=\frac{1}{4}\beta,\ \alpha_2=\frac{1}{4}\alpha_H,\ \beta_2=\frac{1}{8}\beta_H$. The shapes of other solution functions follow similarly noting from above and (4.6) that

$$f_2=g-f_3,\ f_5+f_6=g,\ f_5-f_6=2(w-k),$$

and then using (4.16) and (4.17).

II. The proof proceeds much the same as that of part I. Using $f_1=f_4,\ f_2+f_3=f_5+f_6$, substituting $g=2f_1=2f_4=f_2+f_6=f_3+f_5$ in (2.6) and simplifying, we see that g satisfies the the Sahoo-Székelyhidi's functional equation. Using symmetry, we deduce that it must be of the form (4.19). Thus,

$$\begin{aligned}&2f_1=2f_4=f_2+f_6=f_3+f_5=g\\ &=B(x,y)+\{\phi+\psi(x)+\psi(y)\}+\{\chi(-x)-\chi(x)+\chi(x-y)\}.\end{aligned}$$

The shapes of the solution functions follow by using (4.18), (4.16) and (4.17).

III. Using $f_1=f_4,\ f_2+f_3=f_5+f_6$, substituting $g=2f_1=2f_4=f_2+f_5=f_3+f_6$ in (2.6) and simplifying, we see that g satisfies the Sahoo-Székelyhidi's functional equation. Using symmetry, we deduce that its solution must be of the form (4.19). Thus,

$$\begin{aligned}&2f_1=2f_4=f_2+f_5=f_3+f_6=g\\ &=B(x,y)+\{\phi+\psi(x)+\psi(y)\}+\{\chi(-x)-\chi(x)+\chi(x-y)\}.\end{aligned}$$

The shapes of the solution functions follow by using (4.18), (4.16) and (4.17).

IV. Solving for f_2,f_3,f_5 in terms of f_6 in (4.18) and (4.16), we get

$$f_2=f_6-2k,\ f_3=f_6+2w,\ f_5=f_6+2(w-k).$$

Substituting these functions in (PDF), using (4.16) and (4.17), we get

$$\begin{aligned}&f_1(x+t,y+t)+f_6(x-t,y)+\frac{1}{2}\alpha_K(x+y-t)+f_6(x,y-t)-\frac{1}{2}\alpha_H(x+y-t)\\ &=f_4(x-t,y-t)+f_6(x+t,y)-\frac{1}{2}\alpha_H(x+y+t)\\ &\quad+\frac{1}{2}\alpha_K(x+y+t)+f_6(x,y+t). \qquad (4.20)\end{aligned}$$

From Lemma 4.3, the condition $K(x,x)-H(x,x)=c$ leads to

$$\alpha_K(2x)+\beta-(\alpha_H(2x)+\beta_H(0))=c,$$

i.e., the function $\alpha_K - \alpha_H$ is constant. Using this information and the hypotheses $f_1 = f_4 = f_6$, the equation (4.20) becomes the Sahoo-Székelyhidi's functional equation. Using symmetry, we deduce that its solution must be of the form (4.19). Thus,

$$f_1 = f_4 = f_6 = B(x,y) + \{\phi + \psi(x) + \psi(y)\} + \{\chi(-x) - \chi(x) + \chi(x-y)\}\,.$$

The shapes of the solution functions follow by using (4.18), (4.16) and (4.17). □

5. Skew-symmetric solutions

In this section, we consider solutions of (PDF) which are skew-symmetric functions.

Lemma 5.1. *Let $B\colon G \times G \to \mathbb{C}$ be biadditive and let $\phi, \psi, \chi\colon G \to \mathbb{C}$ be arbitrary functions. If*

$$f(x,y) = B(x,y) + \phi(x) + \psi(y) + \chi(x-y) \tag{5.1}$$

is skew-symmetric, then

$$f(x,y) = B(x,y) - B(x,x) - \psi(x) + \psi(y) + \chi(x-y) - \Phi,$$

where $\Phi = \chi(0)$ is a constant and

$$\chi(x) + \chi(-x) = 2\Phi + B(x,x).$$

Proof. Since $f(x,y)$ is skew-symmetric, equating $f(x,y) = -f(y,x)$, we get

$$B(x,y) + \phi(x) + \psi(y) + \chi(x-y) = -B(y,x) - \phi(y) - \psi(x) - \chi(y-x). \tag{5.2}$$

Substituting $y = 0$, using $B(x,0) = 0 = B(0,x)$, putting $\Phi = -\phi(0) - \psi(0)$ and simplifying, we have

$$\phi(x) = \Phi - \psi(x) - \chi(-x) - \chi(x). \tag{5.3}$$

Replacing this $\phi(x)$ in (5.2) and simplifying we get

$$B(x,y) - \chi(-x) - \chi(x) + \chi(x-y) + 2\Phi = -B(y,x) + \chi(-y) + \chi(y) - \chi(y-x). \tag{5.4}$$

Taking $x = 0$, and simplifying, we get

$$\Phi = \chi(0).$$

Substituting $y = x - z$ and using biadditivity, we get

$$2B(x,x) - B(x,z) - \chi(-x) - \chi(x) + \chi(z) + 2\Phi = B(z,x) + \chi(z-x) + \chi(x-z) - \chi(-z).$$

Replacing z by y, we get

$$2B(x,x) - B(x,y) - \chi(-x) - \chi(x) + \chi(y) + 2\Phi$$

$$= B(y,x) + \chi(y-x) + \chi(x-y) - \chi(-y). \tag{5.5}$$

Combining (5.5) with (5.4) and simplifying, we deduce that

$$\chi(x) + \chi(-x) = 2\Phi + B(x,x). \tag{5.6}$$

From (5.6) and (5.3), we get

$$\phi(x) = -\Phi - \psi(x) - B(x,x). \tag{5.7}$$

Incorporating all the information obtained, the result follows. □

Lemma 5.2. *Let* $f_1, f_2, f_3, f_4, f_5, f_6 \colon G \times G \to \mathbb{C}$*, and let*

$$K(x,y) := f_2 + f_5 - f_3 - f_6, \quad H(x,y) := f_2 - f_5 - f_3 + f_6.$$

If $f_1, f_2, f_3, f_4, f_5, f_6$ *are skew-symmetric functions satisfying* (PDF)*, then*

$$K(x,y) = 0, \quad H(x,y) = -\beta_H(0) + \beta_H(x-y) - 2A_H(x,y), \tag{5.8}$$

where $\alpha_K, \beta_H \colon G \to \mathbb{C}$ *are arbitrary functions,* β *a complex constant,* A_H *a biadditive, skew-symmetric function, and* $\beta_H(t) + \beta_H(-t) = 2\beta_H(0)$*.*

Proof. Using skew-symmetry in (2.2), we get

$$\left(X^tY^t - X^{-t}Y^{-t}\right) S^{(1,4)} + \left(Y^{-t} - Y^t\right) S^{(2,5)} + \left(X^{-t} - X^t\right) S^{(3,6)} = 0. \tag{5.9}$$

Subtracting (5.9) from (2.2) and rearranging, we get

$$\left(X^{-t} - X^t\right) K(x,y) = \left(Y^{-t} - Y^t\right) K(x,y). \tag{5.10}$$

Operating both sides of (5.10) by $X^{-t} - X^t$ and using (5.10) again, we get

$$\begin{aligned}\left(X^{-2t} - 2 + X^{2t}\right) K(x,y) &= \left(Y^{-t} - Y^t\right)\left(X^{-t} - X^t\right) K(x,y)\\ &= \left(Y^{-2t} - 2 + Y^{2t}\right) K(x,y).\end{aligned}$$

Simplifying and replacing $2t$ by t, we have

$$\left(X^{-t} + X^t\right) K(x,y) = \left(Y^{-t} + Y^t\right) K(x,y). \tag{5.11}$$

Defining the function $K_1 \colon G \times G \to \mathbb{C}$ by

$$K_1(x,y) := K\left(\frac{x+y}{2}, \frac{x-y}{2}\right), \text{ or equivalently, } \quad K(x,y) = K_1(x+y, x-y)$$

and rewriting (5.11) in terms of K_1, we get

$$\begin{aligned}&K_1(x+y-t, x-y-t) + K_1(x+y+t, x-y+t)\\ &= K_1(x+y-t, x-y+t) + K_1(x+y+t, x-y-t).\end{aligned}$$

Putting $u = x + y - t,\ v = x - y - t$, this last equation becomes

$$K_1(u,v) + K_1(u+2t, v+2t) = K_1(u, v+2t) + K_1(u+2t, v).$$

Replacing $2t$ by t and rearranging, we get

$$K_1(u+t, v+t) = K_1(u+t, v) + K_1(u, v+t) - K_1(u,v),$$

which is the McKiernan's functional equation mentioned in Proposition 1.1, whose solution is of the form

$$K_1(x,y) = \alpha_K(x) + \beta_K(y) + A_K(x,y).$$

Reverting back to K, we get

$$K(x,y) = \alpha_K(x+y) + \beta_K(x-y) + A_K(x+y, x-y).$$

Since A_K is skew-symmetric and biadditive, the shape of K reduces to

$$K(x,y) = \alpha_K(x+y) + \beta_K(x-y) - 2A_K(x,y). \tag{5.12}$$

Since K is skew-symmetric, solving $K(x,y) = -K(y,x)$, we get

$$2\alpha_K(x+y) + \beta_K(x-y) + \beta_K(y-x) = 0. \tag{5.13}$$

Substituting $x = y$, we get $\alpha_K(x) = -\beta_K(0)$, a constant, and so (5.13) yields

$$\beta_K(t) + \beta_K(-t) = 2\beta_K(0). \tag{5.14}$$

From (5.10) and (5.11), we see at once that $X^t K(x,y) = Y^t K(x,y)$. Using this and (5.12), we get

$$\begin{aligned} &\alpha_K(x+y+t) + \beta_K(x-y+t) - 2A_K(x+t, y) \\ &= \alpha_K(x+y+t) + \beta_K(x-y-t) - 2A_K(x, y+t). \end{aligned}$$

Since A_K is biadditive and skew-symmetric, this last relation simplifies to

$$\beta_K(x-y+t) - \beta_K(x-y-t) = 2A_K(t, x+y).$$

Putting $x = y$, we get $\beta_K(t) - \beta_K(-t) = 2A_K(t, 2x)$, and adding to (5.14), we have

$$\beta_K(t) = \beta_K(0) + A_K(t, 2x).$$

Putting $x = 0$, we see that $\beta_K(t) = \beta_K(0) =: \beta$, a constant, yielding $A_K(x,y) = 0$, and so $K(x,y) = \alpha_K(x+y) + \beta$. Since K is skew-symmetric, equating $K(x,y) = -K(y,x)$, we deduce that $0 = \alpha_K(x+y) + \beta = K(x,y)$.

The determination of H proceeds analogously. Using skew-symmetry in (2.3), we get

$$\left(X^t Y^t + X^{-t} Y^{-t}\right) D^{(1,4)} + \left(Y^{-t} + Y^t\right) D^{(2,5)} + \left(X^{-t} + X^t\right) D^{(3,6)} = 0. \tag{5.15}$$

Subtracting (5.15) from (2.3) and rearranging, we get

$$\left(X^{-t}+X^{t}\right)H(x,y)=\left(Y^{-t}+Y^{t}\right)H(x,y). \tag{5.16}$$

Defining the function $H_1\colon G\times G\to\mathbb{C}$ by

$$H_1(x,y):=H\left(\frac{x+y}{2},\frac{x-y}{2}\right),\quad\text{or equivalently,}\quad H(x,y)=H_1(x+y,x-y)$$

and rewriting (5.16) in terms of H_1, we get

$$\begin{aligned}&H_1(x+y-t,x-y-t)+H_1(x+y+t,x-y+t)\\&=H_1(x+y-t,x-y+t)+H_1(x+y+t,x-y-t).\end{aligned}$$

Putting $u=x+y-t,\ v=x-y-t$, then replacing $2t$ by t and rearranging, we get

$$H_1(u+t,v+t)=H_1(u+t,v)+H_1(u,v+t)-H_1(u,v),$$

which is the McKiernan's functional equation mentioned in Proposition 1.1, whose solution is of the form

$$H_1(x,y)=\alpha_H(x)+\beta_H(y)+A_H(x,y),$$

with the functions α_H,β_H,A_H as stated above. Reverting back to H, we get

$$H(x,y)=\alpha_H(x+y)+\beta_H(x-y)+A_H(x+y,x-y).$$

As in the previous case, using the fact that A_H is skew-symmetric and biadditive, the shape of H reduces to

$$H(x,y)=\alpha_H(x+y)+\beta_H(x-y)-2A_H(x,y).$$

Since H is skew-symmetric, solving $H(x,y)=-H(y,x)$, we get

$$2\alpha_H(x+y)+\beta_H(x-y)+\beta_H(y-x)=0.$$

Substituting $x=y$, we get $\alpha_H(x)=-\beta_H(0)$, a constant and so

$$\beta_H(t)+\beta_H(-t)=2\beta_H(0). \tag{5.17}$$

Incorporating all these details, the shape of H follows. □

Remark 5.1. R1) Setting $t=0$ in (5.15), and simplifying, we get

$$(f_1-f_4)+(f_2-f_5)+(f_3-f_6)=0.$$

R2) From Lemma 5.2 and Lemma 4.2, we have

$$0=K=f_2+f_5-f_3-f_6=-4k(x,y) \tag{5.18}$$

$$-\beta_H(0)+\beta_H(x-y)-2A_H(x,y)=H=f_2-f_5-f_3+f_6=-4w(x,y). \tag{5.19}$$

Throughout the rest of this section, we shall deal only with the case where $f_1 = f_4$, which, by remark R1), gives rise to

$$f_2 - f_5 + f_3 - f_6 = 0.$$

Combining with (5.18) and (5.19), we get

$$f_2 - f_6 = -2k, \ \ f_3 - f_6 = 2w \tag{5.20}$$

Theorem 5.3. *Assume* $f_1, f_2, f_3, f_4, f_5, f_6 \colon G \times G \to \mathbb{C}$ *are skew-symmetric functions satisfying* (PDF).
I. If

$$f_1 = f_4 = \frac{1}{2}\left(f_2 + f_3\right),$$

then there are biadditive function $B(x, y) \colon G \times G \to \mathbb{C}$ *and biadditive, skew-symmetric function* $A(x, y) \colon G \times G \to \mathbb{C}$, *two constants* α_2, Φ, *arbitrary functions* $\psi, \chi, \beta_2 \colon G \to \mathbb{C}$ *with* $\beta_2(t) + \beta_2(-t) = 2\beta_2(0)$ *such that*

$$\begin{aligned}
f_1(x,y) &= f_4(x,y) = B(x,y) - \psi(x) + \psi(y) + \chi(x-y) - \Phi - B(x,x) \\
f_2(x,y) &= f_6(x,y) = f_1(x,y) + \{\alpha_2 + \beta_2(x-y) - A(x,y)\} \\
f_3(x,y) &= f_5(x,y) = f_1(x,y) - \{\alpha_2 + \beta_2(x-y) - A(x,y)\}.
\end{aligned}$$

II. If

$$f_1 = f_4 = \frac{1}{2}\left(f_2 + f_6\right) = \frac{1}{2}\left(f_3 + f_5\right),$$

then there are biadditive function $B(x, y) \colon G \times G \to \mathbb{C}$ *, a constant* Φ, *arbitrary functions* $\psi, \chi \colon G \to \mathbb{C}$ *such that*

$$\begin{aligned}
f_1(x,y) &= f_2(x,y) = f_3(x,y) = f_4(x,y) = f_5(x,y) = f_6(x,y) \\
&= B(x,y) - \psi(x) + \psi(y) + \chi(x-y) - \Phi - B(x,x).
\end{aligned}$$

III. If

$$f_1 = f_4 = \frac{1}{2}\left(f_2 + f_5\right) = \frac{1}{2}\left(f_3 + f_6\right),$$

then there are biadditive function $B(x, y) \colon G \times G \to \mathbb{C}$ *and biadditive, skew-symmetric function* $A(x, y) \colon G \times G \to \mathbb{C}$, *two constants* α_2, Φ, *arbitrary functions* $\psi, \chi, \beta_2 \colon G \to \mathbb{C}$ *with* $\beta_2(t) + \beta_2(-t) = 2\beta_2(0)$ *such that*

$$\begin{aligned}
f_1(x,y) &= f_4(x,y) = B(x,y) - \psi(x) + \psi(y) + \chi(x-y) - \Phi - B(x,x) \\
f_2(x,y) &= f_6(x,y) = f_1(x,y) + \{\alpha_2 + \beta_2(x-y) - A(x,y)\} \\
f_3(x,y) &= f_5(x,y) = f_1(x,y) - \{\alpha_2 + \beta_2(x-y) - A(x,y)\}.
\end{aligned}$$

IV. Assume $f_1 = f_6$ *and*

$$f_1(a,b) + f_6(c,d) + f_6(e,f) = f_1(a',b') + f_6(c',d') + f_6(e',f') \tag{5.21}$$

whenever $a+b+c+d+e+f = a'+b'+c'+d'+e'+f'$. *Then there are biadditive function* $B(x,y)\colon G\times G\to\mathbb{C}$, *a constant* Φ, *arbitrary functions* $\psi,\chi,\beta_2\colon G\to\mathbb{C}$ *with* $\beta_2(t)+\beta_2(-t)=2\beta_2(0)$ *such that*

$$\begin{aligned}f_1(x,y) &= f_2(x,y) = f_4(x,y) = f_6(x,y)\\ &= B(x,y)-\psi(x)+\psi(y)+\chi(x-y)-\Phi-B(x,x)\\ f_3(x,y) &= f_5(x,y) = f_1(x,y)+\beta_2(0)-\beta_2(x-y).\end{aligned}$$

Proof. I. Using $f_1=f_4,\ f_2+f_3=f_5+f_6$, substituting $g=2f_1=2f_4=f_2+f_3=f_5+f_6$ in (2.5) and simplifying, we get

$$\begin{aligned}&g(x+t,y+t)+g(x-t,y)+g(x,y-t)\\ &= g(x-t,y-t)+g(x,y+t)+g(x+t,y),\end{aligned}$$

which is the Sahoo-Székelyhidi's functional equation mentioned in Proposition 1.2, and its solution is of the form

$$g(x,y)=B_1(x,y)+\phi_1(x)+\psi_1(y)+\chi_1(x-y),$$

where $B_1\colon G\times G\to\mathbb{C}$ is biadditive and $\phi_1,\psi_1,\chi_1\colon G\to\mathbb{C}$ are arbitrary functions. Since g is skew-symmetric, applying Lemma 5.1, we deduce that

$$g(x,y)=B_1(x,y)-\psi_1(x)+\psi_1(y)+\chi_1(x-y)-\Phi_1-B_1(x,x), \tag{5.22}$$

where

$$\Phi_1=\chi_1(0),\quad \chi_1(x)+\chi_1(-x)=2\Phi_1+B_1(x,x).$$

Putting

$$B=B_1/2,\ \psi=\psi_1/2,\ \chi=\chi_1/2,\ \Phi=\Phi_1/2,$$

this gives the shapes of f_1 and f_4. Using (4.5), (5.18) and (5.19), we get

$$\begin{aligned}2f_3 &= (f_2+f_3)+(-f_2+f_3)=g+2(w+k)\\ &= g(x,y)-\frac{1}{4}\left\{-\beta_H(0)+\beta_H(x-y)\right\}+\frac{1}{2}A_H(x,y)-0\end{aligned}$$

and the shape of f_3 follows by putting $\alpha_2=-\frac{1}{4}\beta_H(0),\ \beta_2=\frac{1}{4}\beta_H,\ A=\frac{1}{2}A_H$. The shapes of other solution functions follow similarly noting from above and (4.6) that

$$f_2=g-f_3,\ f_5+f_6=g,\ f_5-f_6=2(w-k),$$

and then using (5.18) and (5.19).

II. The proof proceeds much the same as that of part I. Using $f_1=f_4,\ f_2+f_3=f_5+f_6$, substituting $g=2f_1=2f_4=f_2+f_6=f_3+f_5$ in (2.6) and simplifying, we see that g satisfies the the Sahoo-Székelyhidi's functional equation. Using skew-symmetry, we deduce that it must be of the form (5.22). Thus,

$$2f_1=2f_4=f_2+f_6=f_3+f_5=g$$

$$= B_1(x,y) - \psi_1(x) + \psi_1(y) + \chi_1(x-y) - \Phi_1 - B_1(x,x).$$

The shapes of the solution functions follow by using (5.20), (4.16) and (5.19).

III. Using $f_1 = f_4,\ f_2 + f_3 = f_5 + f_6$, substituting $g = 2f_1 = 2f_4 = f_2 + f_5 = f_3 + f_6$ in (2.6) and simplifying, we see that g satisfies the Sahoo-Székelyhidi's functional equation. Using skew-symmetry, we deduce that its solution must be of the form (5.22). Thus,

$$\begin{aligned} &2f_1 = 2f_4 = f_2 + f_5 = f_3 + f_6 = g \\ &= B_1(x,y) - \psi_1(x) + \psi_1(y) + \chi_1(x-y) - \Phi_1 - B_1(x,x). \end{aligned}$$

The shapes of the solution functions follow by using (5.20), (5.18) and (5.19).

IV. Solving for f_2, f_3, f_5 in terms of f_6 in (5.20) and (5.18), we get

$$f_2 = f_6 - 2k,\ f_3 = f_6 + 2w,\ f_5 = f_6 + 2(w-k).$$

Substituting these functions in (PDF), using (5.18) and (5.19), we get

$$\begin{aligned} &f_1(x+t,y+t) + f_6(x-t,y) + f_6(x,y-t) - A_H(x,t) \\ &= f_4(x-t,y-t) + f_6(x+t,y) + f_6(x,y+t) + A_H(t,y). \end{aligned} \tag{5.23}$$

Substituting $t = 0$ in (5.23), we get

$$\begin{aligned} &f_1(x,y) + f_6(x,y) + f_6(x,y) \\ &\qquad = f_4(x,y) + f_6(x,y) + f_6(x,y), \end{aligned}$$

and so $f_1(x,y) = f_4(x,y)$. Substituting $x = 0$ in (5.23), we get

$$\begin{aligned} &f_1(t,y+t) + f_6(-t,y) + f_6(0,y-t) \\ &\qquad = f_4(-t,y-t) + f_6(t,y) + f_6(0,y+t) + A_H(t,y). \end{aligned}$$

Appealing to (5.21), we get $A_H(x,y) = 0$. Substituting $A_H(x,y) = 0$ in (5.23), we get

$$\begin{aligned} &f_1(x+t,y+t) + f_6(x-t,y) + f_6(x,y-t) \\ &\qquad = f_4(x-t,y-t) + f_6(x+t,y) + f_6(x,y+t). \end{aligned}$$

Using $f_1 = f_4 = f_6$, this last relation is the Sahoo-Székelyhidi's functional equation, and so its solution is the form

$$f_1 = f_4 = f_6 = B(x,y) - \psi(x) + \psi(y) + \chi(x-y) - \Phi - B(x,x).$$

Using (5.20), (5.18) and (5.19), we get

$$\begin{aligned} &f_2 = f_6 = B(x,y) - \psi(x) + \psi(y) + \chi(x-y) - \Phi - B(x,x) \\ &f_3 = f_6 + 2w = f_6 + \frac{1}{2}\beta_H(0) - \frac{1}{2}\beta_H(x-y). \end{aligned}$$

Putting $\beta_2 = \beta_H/2$ and observing from (5.17) that $\beta_2(t) + \beta_2(-t) = -2\alpha_2$, the shape of f_3 follows. The remaining functions are $f_5 = f_2 - f_6 + f_3 = f_3$. □

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