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# On the Mersenne sequence

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**Abstract**

From Binet's formula of Mersenne sequence we get some properties for this sequence. Mersenne, Jacobsthal and Jacobsthal-Lucas sequences are considered in order to achieve some relations between them, sums and certain products involving terms of these sequences. We also present some results with matrices involving Mersenne numbers such as the generating matrix, tridiagonal matrices and circulant type matrices.

*Keywords:* Mersenne numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, *g-circulant* matrix.

*MSC:* 11B37, 15B36, 40C05.

## 1. Introduction

In this paper we consider one of the sequences of positive integers satisfying a recurrence relation and we give some well-known identities for this type of sequences. One of the sequences of positive integers (also defined recursively) that have been studied over several years is the well-known Fibonacci (and Lucas) sequence. Many papers are dedicated to Fibonacci sequence, such as the works of Hoggatt, in [15], Vorobiov, in [38], and Koshy, in [32], among others. Others sequences satisfying

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a second-order recurrence relations are the main topic of the research for several authors, such as the studies of the sequences $\{J_n\}_n$ $(n \geq 0)$ and $\{j_n\}_n$ $(n \geq 0)$ of Jacobsthal and Jacobsthal-Lucas numbers, respectively. Recall that the second-order recurrence relation and the initial conditions for the Jacobsthal numbers, $J_n$, $n \geq 0$, and for the Jacobsthal-Lucas numbers, $j_n$, $n \geq 0$, are given by

$$J_{n+2} = J_{n+1} + 2J_n, \; J_0 = 0, \; J_1 = 1 \tag{1.1}$$

and

$$j_{n+2} = j_{n+1} + 2j_n, \; j_0 = 2, \; j_1 = 1, \tag{1.2}$$

respectively. The Binet's formulas for these sequences are given by

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.3}$$

and

$$j_n = 2^n + (-1)^n, \tag{1.4}$$

respectively, where $-1$ and $2$ are the roots of the characteristic equation associated with the above recurrence relations (1.1) and (1.2).

More recently, some of this type of sequences were generalized for any positive real number $k$. The studies of $k$-Fibonacci, $k$-Lucas, $k$-Pell, $k$-Pell-Lucas, Modified $k$-Pell, $k$-Jacobhstal and $k$-Jacobhstal-Lucas sequences are examples of this generalization and some of these studies can be found, in [3–5, 7–9, 16]. More generalizations can be found in [20, 22], where the authors considered the generalized order-$k$ Fibonacci-Pell sequences and the Fibonacci and Lucas $p$-numbers for any given natural number $p$.

In this paper we do not have such kind of generalization, but we will follow closely some of these studies. About the Mersenne sequence, also some studies about this sequence have been published, such as the work of Koshy and Gao in [34], where the authors investigate some divisibility properties of Catalan numbers with Mersenne numbers $M_k$ as their subscripts, developing their work in [33]. In number theory, recall that a Mersenne number of order $n$, denoted by $M_n$, is a number of the form $2^n - 1$, where $n$ is a nonnegative number. This identity is called as the Binet formula for Mersenne sequence and it comes from the fact that the Mersenne numbers can also be defined recursively by

$$M_{n+1} = 2M_n + 1, \tag{1.5}$$

with initial conditions $M_0 = 0$ and $M_1 = 1$. Since this recurrence is inhomogeneous, substituting $n$ by $n + 1$, we obtain the new form

$$M_{n+2} = 2M_{n+1} + 1. \tag{1.6}$$

Subtracting (1.5) to (1.6), we have that $M_{n+2} - M_{n+1} = 2M_{n+1} + 1 - 2M_n - 1$ and then

$$M_{n+2} = 3M_{n+1} - 2M_n, \tag{1.7}$$

other form for the recurrence relation of Mersenne sequence, with initial conditions $M_0 = 0$ and $M_1 = 1$. The roots of the respective characteristic equation $r^2-3r+2 = 0$ are $r_1 = 2$ and $r_2 = 1$ and we easily get the Binet formula

$$M_n = 2^n - 1, \tag{1.8}$$

already given before. Note that there are Mersenne numbers prime and not prime and the search for Mersenne primes is an active field in number theory and computer science. It is now known that for $M_n$ to be prime, $n$ must be a prime $p$, though not all $M_p$ are prime.

There are large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case

$$\begin{aligned}\{M_n\} &= \{0, 1, 3, 7, 15, 31, 63, 127, 255, \ldots\} : A000225\\ \{J_n\} &= \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots\} : A001045\\ \{j_n\} &= \{2, 1, 5, 7, 17, 31, 65, 127, 257, \ldots\} : A014551.\end{aligned}$$

The main purpose of this paper is to present some results involving the Mersenne sequence as a consequence of the respective Binet formula. Besides some relations between Mersenne, Jacobsthal and Jacobsthal-Lucas numbers allows us to obtain some new properties involving sums and products of terms of these sequences. We also present some results with matrices involving Mersenne numbers such as the generating matrix, tridiagonal matrices and circulant type matrices.

## 2. Mersenne sequence and some identities

According with the Binet formula (1.8) for the sequence $\{M_n\}$ we get for this sequence the following interesting identity

**Proposition 2.1** (Catalan's identity)**.** *For $n \geq r$ we have*

$$M_{n-r}M_{n+r} - M_n^2 = 2^{n+1} - 2^{n-r} - 2^{n+r}. \tag{2.1}$$

*Proof.* Using the Binet formula (1.8) we easily obtain

$$\begin{aligned}M_{n-r}M_{n+r} - M_n^2 &= (2^{n-r} - 1)(2^{n+r} - 1) - (2^n - 1)^2\\ &= 2^{2n} - 2^{n-r} - 2^{n+r} + 1 - 2^{2n} + 2^{n+1} - 1\\ &= 2^{n+1} - 2^{n-r} - 2^{n+r},\end{aligned}$$

as required. □

Note that for $r = 1$ in Catalan's identity obtained, we get the Cassini identity for this sequence. In fact, the equation (2.1), for $r = 1$, yields

$$M_{n-1}M_{n+1} - M_n^2 = 2^{n+1} - 2^{n-1} - 2^{n+1}$$

and we get the following result

**Proposition 2.2** (Cassini's identity). *For the sequence* $\{M_n\}_n$ *we have*

$$M_{n-1}M_{n+1} - M_n^2 = -2^{n-1}. \tag{2.2}$$

The d'Ocagne identity for this sequence can also be obtained by the use of the respective Binet formula. Hence, in this case, we get

**Proposition 2.3** (d'Ocagne's identity). *For the sequence* $\{M_n\}_n$ *if* $m > n$, *we have*

$$M_m M_{n+1} - M_{m+1}M_n = 2^n M_{m-n}. \tag{2.3}$$

*Proof.* Once more, using the Binet formula (1.8), we get

$$\begin{aligned} M_m M_{n+1} - M_{m+1}M_n &= (2^m - 1)(2^{n+1} - 1) - (2^{m+1} - 1)(2^n - 1) \\ &= (2^{m+1} - 2^{n+1}) - (2^m - 2^n) \\ &= (2-1)(2^m - 2^n) \\ &= 2^m - 2^n \end{aligned}$$

and the result follows. □

Now we establish some new identities for the common factors of Mersenne numbers and both Jacobsthal and Jacobsthal-Lucas numbers. Such identities are listed in the next propositions and some of these identities involve either Mersenne and Jacobsthal numbers or Mersenne and Jacobsthal-Lucas numbers or all sequences, sums of terms, products of terms, among others.

**Proposition 2.4.** *If* $M_k$, $J_k$ *and* $j_k$ *are the* $k$*th term of the Mersenne sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence, respectively, then*

1. *for an even subscript* $k$,

$$\begin{aligned} 3J_k &= M_k, & (2.4) \\ j_k &= M_k + 2, & (2.5) \end{aligned}$$

2. *and for an odd subscript* $k$

$$\begin{aligned} 3J_k &= M_k + 2, & (2.6) \\ j_k &= M_k. & (2.7) \end{aligned}$$

**Proposition 2.5.** *If* $M_j$ *is the* $j$*th term of the Mersenne sequence then*

1. $M_n^2 = 4^n - M_{n+1}$;
2. $\sum_{j=0}^{n} M_j = M_{n+1} - (n+1) = 2M_n - n$;
3. $$M_n + J_n = \begin{cases} 3J_n + J_n = 4J_n, & \text{if } n \text{ is an even number} \\ 3J_n - 2 + J_n = 4J_n - 2, & \text{if } n \text{ is an odd number} \end{cases}$$

*4.*

$$M_n + j_n = \begin{cases} j_n - 2 + j_n = 2j_n - 2, & \text{if } n \text{ is an even number} \\ 2j_n, & \text{if } n \text{ is an odd number} \end{cases}$$

*5.*

$$M_n \times J_n = \begin{cases} 3J_n^2, & \text{if } n \text{ is an even number} \\ (3J_n - 2)J_n = 3J_n^2 - 2J_n, & \text{if } n \text{ is an odd number} \end{cases}$$

*6.*

$$M_n \times j_n = \begin{cases} (j_n - 2)j_n = j_n^2 - 2j_n, & \text{if } n \text{ is an even number} \\ j_n \times j_n = j_n^2, & \text{if } n \text{ is an odd number} \end{cases}$$

*Proof.* Using the Binet Formula for the Mersenne numbers, the first identity easily follows. For the second one according the equation (1.5) we have

$$\begin{aligned} M_1 &= 2M_0 + 1 \\ M_2 &= 2M_1 + 1 \\ M_3 &= 2M_2 + 1 \\ &\cdots \\ M_{n+1} &= 2M_n + 1. \end{aligned}$$

Thus if we get the sum of both terms of these equations we obtain

$$\sum_{j=1}^{n+1} M_j = 2\sum_{j=0}^{n} M_j + n + 1,$$

which implies (using the fact that $M_0 = 0$) that

$$\begin{aligned} 2\sum_{j=0}^{n} M_j &= \sum_{j=1}^{n+1} M_j - n - 1 \\ &= \sum_{j=0}^{n} M_j - n - 1 - M_0 + M_{n+1} \\ &= \sum_{j=0}^{n} M_j - n - 1 + M_{n+1} \end{aligned}$$

Hence $\sum_{j=0}^{n} M_j = M_{n+1} - (n+1)$ and the result follows by equation (1.5).

About the others identities in this proposition, we decided to omit its proof here because they can be easily proven. □

Again using the Binet formula (1.8), we obtain other property of the Mersenne sequence which is stated in the following proposition.

**Proposition 2.6.** *If $M_n$ is the nth term of the Mersenne sequence, then we have*

$$\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = r_1. \tag{2.8}$$

*Proof.* We have that

$$\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = \lim_{n\to\infty} \left(\frac{2^n - 1}{2^{n-1} - 1}\right) = \lim_{n\to\infty} \left(\frac{1 - \frac{1}{2^n}}{\frac{1}{2} - \frac{1}{2^n}}\right). \tag{2.9}$$

Since $\left|\frac{1}{2}\right| < 1$, $\lim_{n\to\infty}(\frac{1}{2})^n = 0$. Next we use this fact in (2.9) obtaining

$$\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = \frac{1}{\frac{1}{2}} = r_1. \qquad \square$$

Also, we easily can show the following result using basic tools of calculus of limits and (2.8).

**Corollary 2.7.** *If $M_n$ is the nth term of the Mersenne sequence, then*

$$\lim_{n\to\infty} \frac{M_{n-1}}{M_n} = \frac{1}{r_1}. \tag{2.10}$$

# 3. Matrices with Mersenne numbers

## 3.1. Generating Matrix

One of the usual methods for the study of the recurrence sequences is the use of a generating matrix. The Mersenne numbers form a sequence of such type which is defined recursively as a linear combination of the preceding $p$ terms

$$a_{n+p} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{p-1} a_{n+p-1}, \tag{3.1}$$

where $c_0, c_1, \ldots, c_{p-1}$ are real constants. Some authors used the matrix method for the case of the study of some recurrence sequences of natural numbers (see, for example, [2, 6, 19–21, 23, 25], among others). Consider a square matrix $M$ of order $p \times p$ such that the least line is $c_0, c_1, \ldots, c_{p-1}$, the $(i, i+1)-$entries, for $i = 1, \ldots, p-1$, are equal to 1 and the remaining entries are zero. According to (1.7) and (3.1) we have that $p = 2$, $c_0 = -2$ and $c_1 = 3$. Hence the matrix $M$ is given by

$$M = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \tag{3.2}$$

with $|M| = 2$.

**Proposition 3.1.** *For $n \geq 1$ we have*

$$M^n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix}, \tag{3.3}$$

*Proof.* We use induction on $n$. For $n = 1$,

$$M = \begin{pmatrix} -2M_0 & M_1 \\ -2M_1 & M_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix},$$

which is verified by (1.7) taking into account the initial conditions. Suppose that (3.3) is valid for $n$. Then

$$\begin{aligned} M^{n+1} = MM^n &= \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} -2M_n & M_{n+1} \\ 4M_{n-1} - 6M_n & -2M_n + 3M_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} -2M_n & M_{n+1} \\ -2M_{n+1} & M_{n+2} \end{pmatrix}, \end{aligned}$$

as required. □

Using this proposition and some properties involving the determinant of the matrices $M$ and $M^n$, we obtain Cassini's identity in a different way than Proposition 2.2.

## 3.2. Special kind of tridiagonal matrix

We use tridiagonal matrices in the same way that Falcon used in [12]. A tridiagonal matrix is a matrix that has nonzero elements only on the main diagonal and on the first diagonal below and above this. Let $A_n$ a square matrix ($n \geq 1$) of order $n \geq 1$ defined by

$$A_n = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & d & e & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & d & e & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & d & e \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & d \end{pmatrix}, a, b, c, d, e \in \mathbb{R}.$$

Using some properties of the determinant of a tridiagonal matrix of order $n$, we have

$$\begin{aligned} |A_1| &= a \\ |A_2| &= d|A_1| - bc \\ |A_3| &= d|A_2| - ce|A_1| \\ &\vdots \\ |A_{n+1}| &= d|A_n| - ce|A_{n-1}|. \end{aligned}$$

If $a = 3$, $b = 2$, $c = -1$, $d = 3$ and $e = -2$, the matrix $A_n$ is the tridiagonal matrix:

$$N_n = \begin{pmatrix} 3 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{pmatrix}, \tag{3.4}$$

In this case,

$$\begin{aligned} |N_1| &= 3 = M_2 \\ |N_2| &= 3|N_1| - 2 \times (-1) = 7 = M_3 \\ |N_3| &= 3|N_2| - (-1) \times (-2)|N_1| = 15 = M_4 \\ &\vdots \\ |N_{n+1}| &= 3|N_n| - (-1) \times (-2)|N_{n-1}| = 3|N_n| - 2|N_{n-1}|. \end{aligned}$$

for $n \geq 1$. Then we have the following result involving the Mersenne number of order $n$ in terms of the determinant of a tridiagonal matrix:

**Proposition 3.2.** *If $N_n$ is the $n$–by–$n$ tridiagonal matrix considered in* (3.4)*, then the $n$th Mersenne number is given by $M_n = |N_{n-1}|$.*

Now, using the tridiagonal matrix (2.3) considered in [31] for any recurrence sequence of order two, we have that $C = M_0$, $D = M_1$, $A = 3$ and $B = -2$. Note that $A$ and $B$ are such that

$$x_{n+1} = Ax_n + Bx_{n-1}, n \geq 1$$

with $C = x_0$, $D = x_1$ and $\{x_n\}$ a sequence defined by this recurrence relation of order 2. The corresponding tridiagonal matrix considered in [31] for the Mersenne sequence is

$$N'_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & -2 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -2 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} |N'_0| &= 0 = M_0 \\ |N'_1| &= \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 = M_1 \end{aligned}$$

$$|N_2'| = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 3 \end{vmatrix} = 3 = M_2$$

$$\vdots$$

and then

**Proposition 3.3.** *Let $N_n'$ is the $(n+1)-by-(n+1)$ tridiagonal matrix considered in* (3.5)*, then the nth Mersenne number is given by $M_n = |N_n'|$.*

Some authors studied the relationships between determinant or permanent of certain tridiagonal matrices and terms of recursive sequences, such as the works of Kiliç and Stanica in [24], Kiliç and Taşci in [26], [27] and [30]. Next we derive relationships between the Mersenne numbers and determinant or permanent of certain Hessenberg matrices as complementary of the results that we have already obtained. First we stated some background related with it.

The *permanent* of an $n$-square matrix $A = (a_{ij})$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

A matrix is said to be a $(0,1)$-matrix if each of its entries are 0 or 1.

In [28], the authors consider the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs. In this paper, the authors consider families of square matrices such that each matrix is the adjacency matrix of a bipartite graph and the permanent of the matrix is a sum of consecutive Fibonacci or Lucas numbers. For Fibonacci and Lucas sequences, they consider special $n$-square $(0,1)$-matrices for which the sum of two of these matrices - one tridiagonal matrix and the other not tridiagonal - is the adjacency matrix of a bipartite graph and its permanent is a sum of consecutive elements of these sequences. Also, in [22], the authors consider certain generalizations of the Fibonacci and Lucas $p$-numbers. In this paper are considered certain families of square matrices such that each matrix is the adjacency matrix of a bipartite graph which its permanent is the generalized Fibonacci or Lucas $p$-numbers and a sum of consecutive elements of these sequences. Also, the authors consider certain matrices whose determinants are related with the Fibonacci and Lucas $p$-numbers and their sums. In [29] some relationships between the sums of second order linear recurrences and permanent or determinants of certain Hessenberg matrices are derived. Motivated essentially by these works, next we present some more results involving the permanent, the determinant of certain Hessenberg matrices and the sums of Mersenne numbers. Following the work of Kiliç and Taşci in [29] and [30] and if we consider the matrix

$T_n = (t_{ij})$ defined by

$$T_n = \begin{pmatrix} r_1 + r_2 & r_2 & 0 & \cdots & 0 \\ r_1 & r_1 + r_2 & r_2 & \cdots & 0 \\ 0 & r_1 & r_1 + r_2 & r_2 & \vdots \\ \vdots & \vdots & \vdots & \ddots & r_2 \\ 0 & 0 & \cdots & r_1 & r_1 + r_2 \end{pmatrix}, \tag{3.6}$$

where $r_1$ and $r_2$ are the roots of the characteristic equation associated to identity (1.7). According to the values of $r_1$ and $r_2$ and the result (1.8) in [29] we obtain, without any proof, the relationship between the determinant of a certain Hessenberg matrix and Mersenne numbers.

**Proposition 3.4.** *If $M_j$ is the $j$th term of the Mersenne sequence and $R_n$ is the matrix $T_n$ in (3.6) considering the values of $r_1$ and $r_2$ for Mersenne sequence, then*

$$det R_n = M_{n+1}.$$

Next, using Theorems 1, 2 and 3 of [29] applied to the Mersenne numbers, we immediately get the relationship between the permanent of a certain tridiagonal matrix and Mersenne numbers. Note that the last part of the following proposition relates the determinant of a certain matrix with the sum of Mersenne numbers and for the last two parts we use the result obtained in Section 2, namely in item 2 of Proposition 2.5.

**Proposition 3.5.** *Let $M_j$ be the $j$th term of the Mersenne sequence and let $n$ be a natural number. If*

*1.*

$$A_n = \begin{pmatrix} 3 & -2 & 0 & \cdots & 0 \\ 1 & 3 & -2 & \cdots & 0 \\ 0 & 1 & 3 & -2 & \vdots \\ \vdots & \vdots & \vdots & \ddots & -2 \\ 0 & 0 & \cdots & 1 & 3 \end{pmatrix}, \tag{3.7}$$

*then $perm A_n = M_{n+1}$;*

*2.*

$$H_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & \\ 0 & & A_n & & \\ \vdots & & & & \end{pmatrix}, \tag{3.8}$$

*then $perm H_n = \sum_{i=0}^{n} M_i = 2M_n - n$;*

*3.*

$$G_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 3 & -2 & \cdots & 0 \\ 0 & -1 & 3 & -2 & \vdots \\ \vdots & \vdots & \vdots & \ddots & -2 \\ 0 & 0 & \cdots & -1 & 3 \end{pmatrix}, \tag{3.9}$$

*then* $detG_n = \sum_{i=0}^{n} M_i = 2M_n - n.$

## 3.3. Circulant type matrix

Circulant matrices have been a great topic of research and its history and applications are vast (see, for example, [10, 14, 17, 37, 40]). All types of circulant matrices arise in the study of periodic or multiply symmetric dynamical systems and they play a crucial role for solving various differential equations (see, for example, [1, 11, 35, 39]). These matrices have been exploited to obtain the transient solution in closed form for fractional order differential equations (see, for example, [1]). Wu and Zou in [39] discussed the existence and approximation of solutions of asymptotic or periodic boundary value problems of mixed functional differential equations. In the literature, papers on several types of circulant matrices have been published (see, for example, [36]). Some authors study these type of matrices whose entries are integers that belong to sequences defined recursively. This is, for example the case of [36] and [41] where the authors considered circulant matrices with the Fibonacci and Lucas numbers and in [13] where are considered circulant matrices whose entries are Jacobsthal and Jacobsthal-Lucas numbers.

For a natural number $n$ and a nonnegative integer $g$, a *g-circulant* matrix is a square matrix of order $n$ with the following form:

$$A_{g,n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix}, \tag{3.10}$$

where each of the subscripts is understood to be reduced modulo $n$. The first row of $A_{g,n}$ is $(a_1, a_2, \ldots, a_n)$ and its $(j+1)$th row is obtained by giving its $j$th row a right circular shift by $g$ positions.

If $g = 1$ or $g = n + 1$ we obtain the standard *right circulant* matrix, or simply, *circulant* matrix. Thus a right circulant matrix is written as

$$RCirc(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}. \tag{3.11}$$

If $g = n-1$, we obtain the standard *left circulant* matrix, or *reverse circulant* matrix. In this case we write a left circulant matrix as

$$LCirc(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{pmatrix}. \tag{3.12}$$

Let $A_n = RCirc(M_1, M_2, \ldots, M_n)$ be a *right circulant* matrix. Using the idea of Gong, Jiang and Gao in [13], we present a determinant formula for $A_n$.

**Theorem 3.6.** *For $n \geq 1$, let $A_n = RCirc(M_1, M_2, \ldots, M_n)$ be a right circulant matrix. Then we have*

$$detA_n = (1 - M_{n+1})^{n-1} + (-2M_n)^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - M_{n+1}}{-2M_n} \right)^{k-1} (-2M_k). \tag{3.13}$$

*Proof.* For $n = 1$, $detA_1 = M_1$ satisfies (3.13). In the case of $n \geq 2$, we consider the square matrices of order $n$ of common use in the theory of circulant matrices

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -3 & 2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left( \frac{-2M_n}{1-M_{n+1}} \right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left( \frac{-2M_n}{1-M_{n+1}} \right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{-2M_n}{1-M_{n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that

$$det\Gamma = det\Pi = (-1)^{\frac{(n-1)(n-2)}{2}}. \tag{3.14}$$

Calculating the product $\Gamma A_n \Pi$ we obtain

$$C = \begin{pmatrix} M_1 & \delta_n & M_{n-1} & \cdots & M_3 & M_2 \\ 0 & \delta'_n & M_{n-2} & \cdots & -2M_2 & -2M_1 \\ 0 & 0 & M_1 - M_{n+1} & & & \\ 0 & 0 & 2M_n & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & 2M_n & M_1 - M_{n+1} \end{pmatrix},$$

where

$$\delta_n = \sum_{k=1}^{n-1} \left( \frac{2M_n}{M_1 - M_{n+1}} \right)^{n-(k+1)} M_{k+1}$$

and

$$\delta'_n = (M_1 - M_{n+1}) + \sum_{k=1}^{n-1} \left( \frac{-2M_n}{M_1 - M_{n+1}} \right)^{n-(k+1)} (-2M_k). \quad (3.15)$$

Calculating the determinant of the matrix $C = \Gamma A_n \Pi$ we obtain

$$detC = M_1 \delta'_n (M_1 - M_{n+1})^{n-2}. \quad (3.16)$$

Using the identity (3.15), the recurrence relation (1.7) with the initial condition ($M_1 = 1$) and doing some calculations, a new expression for the determinant (3.16) is given by

$$detC = (1 - M_{n+1})^{n-1} + (-2M_n)^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - M_{n+1}}{-2M_n} \right)^{k-1} (-2M_k). \quad (3.17)$$

Using the property of the determinant of a product of matrices and the identity (3.14), we conclude that

$$detA_n = detC$$

and the result follows. □

Let $B_n = LCirc(M_1, M_2, \ldots, M_n)$ be a *left circulant* matrix whose entries are Mersenne numbers. We present a determinant formula for $B_n$ using again the idea of Gong, Jiang and Gao in [13]. Lemma 5 in [18] will help us to obtain this formula. In Lemma 5 of [18] the authors define a matrix $\Delta$ which is an orthogonal cyclic shift matrix (and a left circulant matrix) of order $n$. They stated that

$$LCirc(a_1, a_2, \ldots, a_n) = \Delta RCirc(a_1, a_2, \ldots, a_n). \quad (3.18)$$

Using the fact that $det\Delta = (-1)^{\frac{(n-1)(n-2)}{2}}$, calculating the determinant in both sides of the identity (3.18) and according to the result obtained in Theorem 3.6, the following result is easily proved.

**Theorem 3.7.** *For $n \geq 1$, let $B_n = LCirc(M_1, M_2, \ldots, M_n)$ be a left circulant matrix. Then we have*

$$detB_n =$$
$$(-1)^{\frac{(n-1)(n-2)}{2}} \left( (1 - M_{n+1})^{n-1} + (-2M_n)^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - M_{n+1}}{-2M_n} \right)^{k-1} (-2M_k) \right).$$

Let us consider $C_n = A_{g,n}$ be a $g$-circulant matrix defined as in (3.10), whose entries are Mersenne numbers. We present the determinant formula of $C_n$ and for that we use Lemma 6 and Lemma 7 of [18]. Thus, from these Lemmas and Theorem 3.6, we deduce the following result

**Theorem 3.8.** *Let $C_n = A_{g,n}$ be a $g$-circulant matrix defined as in (3.10), whose entries are Mersenne numbers. Then one has*

$$detC_n = det\mathbb{Q}_g[(1 - M_{n+1})^{n-1} + (-2M_n)^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - M_{n+1}}{-2M_n} \right)^{k-1} (-2M_k)],$$

*where $\mathbb{Q}_g$ is a $g$-circulant matrix with the first row $e^* = [1, 0, \ldots, 0]$.*

# 4. Conclusions

Sequences of integer numbers have been studied over several years, with emphasis on studies of the well known Fibonacci sequence (and then the Lucas sequence) that is related to the golden ratio and of the Pell sequence that is related to the silver ratio. In this paper we also contribute for the study of Mersenne sequence giving some identities which some of them involve Jacobsthal and Jacobsthal-Lucas numbers.

Several studies involving all types of circulant matrices and tridiagonal matrices can easily be found in the literature. Here we have considered the $g$-circulant, right and left circulant matrices whose entries are Mersenne numbers. For these cases we have provided the determinant of these matrices.

In the future, we intend to discuss the invertibility of these circulant type matrices associated with these sequence, such as Shen, in [36], did in the case of Fibonacci and Lucas numbers.

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