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Statistics in words and partitions of a set

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Abstract

Let $[k] = \{1, 2, \ldots, k\}$ be an alphabet over k letters. A *word* ω of length n over alphabet $[k]$ is an element of $[k]^n$ and is also called a k-ary word of length n. We say that ω contains an ℓ-peak, if it exists an i such that $2 \leq i \leq n - \ell$ where $\omega_i = \omega_{i+1} = \cdots = \omega_{i+\ell-1}$ and $\omega_{i-1} < \omega_i$ and $\omega_{i+\ell-1} > \omega_{i+\ell}$. A *partition* Π of set $[n]$ of size k is a collection $\{B_1, B_2, \ldots, B_k\}$ of non empty disjoint subsets of $[n]$, called *blocks*, whose union equals $[n]$. In this paper, we find an explicit formula for the generating function for the number of words of length n over alphabet $[k]$ according to the number of ℓ-peaks in terms of Chebyshev polynomials of the second kind. As a consequence of the results obtained for words, we finally find the number of ℓ-peaks in set partitions of $[n]$ with exactly k blocks.

Keywords: Set partitions, words, ℓ-peak, Chebyshev polynomials of the second kind

MSC: 05A05

1. Introduction

Words

Let $[k] = \{1, 2, \ldots, k\}$ be an alphabet over k letters. A *word* ω of length n over alphabet $[k]$ is an element of $[k]^n$ and is also called a word of length n on k letters or a k-ary word of length n. The number of the words of length n over alphabet $[k]$ is k^n. Similar statistics in patterns of subwords have been widely studied in the literature (see [2]). For example, Kitaev, Mansour and Remmel [3] enumerated the number of rises (respectively, levels and falls) which are subword patterns 12, (respectively, 11 and 21) in words that have a prescribe first element. Heubach and Mansour [2] enumerated the number of words of length n over alphabet $[k]$ that contain the subword pattern 111 and the subword pattern 112 exactly r times. Burstein and Mansour [1] generalized the result to subword pattern of length ℓ. More recently, Mansour [4] enumerated the number of peaks (subword patterns 121, 132 or 231) and valleys (subword patterns 212, 213 or 312) in words of length n over alphabet $[k]$. Our aim is to extend this result to patterns of arbitrary length. We say that ω contains an *ℓ-peak*, if exists $2 \leq i \leq n-\ell$ such that $\omega_i = \omega_{i+1} = \cdots = \omega_{i+\ell-1}$ and $\omega_{i-1} < \omega_i$ and $\omega_{i+\ell-1} > \omega_{i+\ell}$. For example, the word $12^4 13^4 2 = 12222133332$ in $[3]^{11}$ contains two 4-peaks, namely 122221 and 133332.

Set partitions

A *partition* Π of set $[n]$ with exactly k blocks is a collection $\{B_1, B_2, \ldots, B_k\}$ of non empty disjoint subsets of $[n]$ whose union is equal to $[n]$. We assume that blocks are listed in increasing order of their minimal elements, that is, $minB_1 < minB_2 < \cdots < minB_k$. We denote the set of all partitions of $[n]$ with exactly k blocks to be $P_{n,k}$. The number of all partitions of $[n]$ with k blocks is $S(n,k)$, these are the Stirling numbers of the second kind [9]. We denote the set of all partitions of $[n]$ to be P_n, namely $P_n = \cup_{k=0}^{n} P_{n,k}$. The number of all partitions of $[n]$ is $B_n = \sum_{k=0}^{n} S_{n,k}$, which is the n-th Bell number. Any partition Π can be written as $\pi_1\pi_2\cdots\pi_n$, where $i \in B_{\pi_i}$ for all i, and this form is called the *canonical sequential form*. For example $\Pi = \{\{12\},\{3\},\{4\}\}$ is a partition of [4], the canonical sequential form is $\pi = 1123$. Several authors have studied different statistics on P_n (see [4]). For instance, Mansour and Munagi [6] found the generating function for the number of partitions of $[n]$ according to rises, descents and levels, they also computed the total number of t-rises (respectively, t-descents and t-levels), this is a increasing subword pattern of size t (respectively, decreasing subword pattern of size t, fixed subword pattern of size t), see [5]. A lot of attention has been given to the statistics on $P_{n,k}$ (see [4]). For example, Shattuck [8] counted the rises, descents and levels in the set partition of $[n]$ with exactly k blocks. In addition, Mansour [4] found an explicit formula for the generating functions for the number of set partition of $[n]$ with exactly k blocks according to the statistics ℓ-rise (respectively, ℓ-descent and ℓ-level). Mansour and Shattuck [7] found an explicit formula for the generating function of set partitions of n with exactly $[k]$ blocks according to the number of peaks (valleys). Our aim is to extend this result for the set $P_{n,k}$ according to the number of ℓ-peaks.

In this paper, we find the generating function of the words of length n over alphabet $[k]$ according to the number of ℓ-peaks. We also compute the total number of ℓ-peaks in the words of length n over alphabet $[k]$. As a consequence of these results, we find the number of ℓ-peaks in set partitions of $[n]$ with exactly k blocks.

2. Words and partitions of a set according to multi statistics ℓ-peaks

Let $W_k(x, q_1, \ldots, q_\ell)$ be the generating function for the number of words of length n over alphabet $[k]$ according to the number of ℓ-peaks, namely,

$$W_k(x, q_1, \ldots, q_\ell) = \sum_{n\geq 0} x^n \sum_{\omega\in[k]^n} \prod_{i=1}^{\ell} q_i^{i-peak(\omega)}.$$

Lemma 2.1. *The generating function $W_k(x, q_1, \ldots, q_\ell)$ satisfies the recurrence relation*

$$W_k(x, q_1, \ldots, q_\ell) = \frac{A_\ell - xB_\ell + W_{k-1}(x, q_1, \ldots, q_\ell)(B_{\ell+1} - A_\ell)}{(1-x)(1+A_\ell) + x^{\ell+1} - W_{k-1}(x, q_1, \ldots, q_\ell)\big((1-x)A_\ell + x^{\ell+1}\big)},$$

where $A_\ell = \sum_{i=1}^{\ell} x^i q_i$ and $B_\ell = \frac{1-x^\ell}{1-x}$.

Proof. It is obvious

$$W_k(x, q_1, \ldots, q_\ell) = W_{k-1}(x, q_1, \ldots, q_\ell) + W_k^\dagger(x, q_1, \ldots, q_\ell), \tag{2.1}$$

where $W_k^\dagger(x, q_1, \ldots, q_\ell)$ is the generating function for the number of words ω of length n over alphabet $[k]$ according to the number of ℓ-peaks such that ω contains at least one occurrence of the letter k. A word ω that contains a letter k can be decomposed as either

(1) k;

(2) $k\omega'$, where ω' is a non empty word over $[k]$;

(3) $\omega'' k^i \omega'''$, where k^i denotes a word $kk\cdots k$ with exactly i letters, ω'' is a non empty word over $[k-1]$ and ω''' is a non empty word over $[k]$ which starts with a letter $a \neq k$, for $1 \leq i \leq \ell$;

(4) $\omega'' k^i$, for $1 \leq i \leq \ell$; or

(5) $\omega'' k^{\ell+1} \omega''''$, where ω'''' is a word over $[k]$.

The corresponding generating functions of these decomposition are

(1) x;

(2) $x(W_k(x, q_1, \ldots, q_\ell) - 1)$;

(3) $q_i x^i (W_{k-1}(x, q_1, \ldots, q_\ell) - 1)(W_k(x, q_1, \ldots, q_\ell)(1-x) - 1)$, for $1 \leq i \leq \ell$;

(4) $x^i (W_{k-1}(x, q_1, \ldots, q_\ell) - 1)$, for $1 \leq i \leq \ell$; or

(5) $x^{\ell+1}(W_{k-1}(x,q_1,\ldots,q_\ell)-1)W_k(x,q_1,\ldots,q_\ell)$,

respectively. Hence, by (2.1), we obtain

$$\begin{aligned}&W_k(x,q_1,\ldots,q_\ell)\\&=W_{k-1}(x,q_1,\ldots,q_\ell)+x+x(W_k(x,q_1,\ldots,q_\ell)-1)\\&\quad+\sum_{i=1}^{\ell}q_ix^i(W_{k-1}(x,q_1,\ldots,q_\ell)-1)(W_k(x,q_1,\ldots,q_\ell)(1-x)-1)\\&\quad+\sum_{i=1}^{\ell}x^i(W_{k-1}(x,q_1,\ldots,q_\ell)-1)\\&\quad+x^{\ell+1}(W_{k-1}(x,q_1,\ldots,q_\ell)-1)W_k(x,q_1,\ldots,q_\ell),\end{aligned}$$

which equivalent to

$$\begin{aligned}&W_k(x,q_1,\ldots,q_\ell)\\&=\frac{A_\ell-xB_\ell+W_{k-1}(x,q_1,\ldots,q_\ell)(B_{\ell+1}-A_\ell)}{(1-x)(1+A_\ell)+x^{\ell+1}-W_{k-1}(x,q_1,\ldots,q_\ell)\left((1-x)A_\ell+x^{\ell+1}\right)},\end{aligned}\tag{2.2}$$

where $A_\ell=\sum_{i=1}^{\ell}x^iq_i$ and $B_\ell=\frac{1-x^\ell}{1-x}$. □

We plan to find an explicit formula for the generating function $P_k(x,q_1,\ldots,q_\ell)$ for the number of partitions of n with exactly k blocks according to the number of ℓ-peaks.

$$P_k(x,q_1,\ldots,q_\ell)=\sum_{n\geq0}x^n\sum_{\pi\in P_{n,k}}\prod_{i=1}^{\ell}q^{i-peak(\pi)}.$$

To do that we will use Lemma 2.1.

Theorem 2.2. *For all $k\geq1$,*

$$\begin{aligned}&P_k(x,q_1,\ldots,q_\ell)\\&=\prod_{j=1}^{k}\left(\sum_{i=1}^{\ell}x^i(q_i(W_j(x,q_1,\ldots,q_\ell)(1-x)-1)+1)+x^{\ell+1}W_j(x,q_1,\ldots,q_\ell)\right).\end{aligned}$$

Proof. Any partition π of $[n]$ with exactly k blocks can be decomposed either

(1) $\pi k^i\pi'$, πk^i, for $1\leq i\leq\ell$, where π is a set partition with exactly $k-1$ blocks, π' is a non empty word over alphabet $[k]$ which starts with a letter $a<k$; or

(2) $\pi k^{\ell+1}\pi''$, where π'' is a word over alphabet $[k]$.

The corresponding generating functions are

(1)

$$q_i x^i P_{k-1}(x,q_1,\dots,q_\ell)(W_k(x,q_1,\dots,q_\ell) - xW_k(x,q_1,\dots,q_\ell)-1) + x^i P_{k-1}(x,q_1,\dots,q_\ell),$$

for $1 \le i \le \ell$;

(2) $x^{\ell+1}P_{k-1}(x,q_1,\dots,q_\ell)W_k(x,q_1,\dots,q_\ell)$,

respectively. By summing all the last terms we obtain

$$\begin{aligned}
&P_k(x,q_1,\dots,q_\ell)\\
&=\sum_{i=1}^{\ell} q_i x^i P_{k-1}(x,q_1,\dots,q_\ell)(W_k(x,q_1,\dots,q_\ell)-xW_k(x,q_1,\dots,q_\ell)-1)\\
&\quad+\sum_{i=1}^{\ell} x^i P_{k-1}(x,q_1,\dots,q_\ell)+x^{\ell+1}P_{k-1}(x,q_1,\dots,q_\ell)W_k(x,q_1,\dots,q_\ell)\\
&=P_{k-1}(x,q_1,\dots,q_\ell)\cdot\\
&\quad\cdot\left(\sum_{i=1}^{\ell} x^i(q_i(W_j(x,q_1,\dots,q_\ell)(1-x)-1)+1)+x^{\ell+1}W_j(x,q_1,\dots,q_\ell)\right).
\end{aligned}$$

Thus, by induction on k together with the initial condition $P_0(x,q)=1$, we complete the proof. □

Example 2.3. Using the recursion given in Theorem 2.2, we may obtain the generating function for the number of partitions of $[n]$ with exactly k blocks,

$$\begin{aligned}
P_k(x,1,\dots,1) &= \prod_{j=1}^{k}\sum_{i=1}^{\ell} x^i W_j(x,1,\dots,1)(1-x)+x^{\ell+1}W_j(x,1,\dots,1)\\
&=\prod_{j=1}^{k}\left(x\frac{1-x^\ell}{1-x}\frac{1}{1-jx}(1-x)+x^{\ell+1}\frac{1}{1-jx}\right)\\
&=x^k\prod_{j=1}^{k}\frac{1}{1-jx},
\end{aligned}$$

which is in accord with the well-known the generating function for the number of partitions of $[n]$ with exactly k blocks.

Example 2.4. By substituting $\ell = 1$ and $q_1 = q$ in Lemma 2.1, we get $W_k(x,q)$ the generating function for the number of words of length n over the alphabet $[k]$ according to the number of peaks (peak of length one), which gives the following recursion

$$W_k(x,q)=\frac{x(q-1)+(1-x(q-1))W_{k-1}(x,q)}{1-x(1-q)(1-x)-x(x+q(1-x))W_{k-1}(x,q)}.$$

By using the same substitution in Theorem 2.2, we obtain the recurrence relation for the generating function for the number of set partitions $P_{n,k}$ according to the number of peaks (peak of length one), which gives the following recursion

$$P_k(x,q) = x^k \prod_{j=1}^{k} (1 + x(1-q)W_j(x,q) + q(W_j(x,q) - 1)),$$

where the two above results agree with the results of Mansour and Shattuck (see [7]).

2.1. Counting ℓ-peaks in words and partitions of a set

Let $W_k(x,q)$ be the generating function for the number of words of length n over alphabet $[k]$ according to the number of ℓ-peaks.

$$W_k(x,q) = \sum_{n\geq 0} x^n \left(\sum_{\omega\in[k]^n} q^{\ell-peak(\omega)} \right).$$

Corollary 2.5. *The generating function $W_k(x,q)$ for the number of words of length n over alphabet $[k]$ according to the number of ℓ-peaks is*

$$W_k(x,q) = \frac{A + (1-A)W_{k-1}(x,q)}{a_\ell - xW_{k-1}(x,q)a_{\ell-1}} \tag{2.3}$$

where $A = x^\ell(q-1)$ and $a_\ell = 1 + x^\ell(q-1)(1-x)$, which is equivalent to

$$W_k(x,q) = \frac{x^\ell(q-1)(U_{k-1}(t) - U_{k-2}(t))}{U_k(t) - U_{k-1}(t) - (1 - x^\ell(q-1))(U_{k-1}(t) - U_{k-2}(t))}, \tag{2.4}$$

where $t = 1 + \frac{x^{\ell+1}}{2}(1-q)$ and U_m is the m-th Chebyshev polynomial of the second kind.

Proof. By substituting $q_i = 1$ for $i \neq \ell$, and $q_\ell = q$ in (2.2) we obtain (2.3). Then, by applying [Appendix D] [4] for (2.2), we obtain (2.4). □

Now, our aim is to find the total number of ℓ-peaks in all words of length n over alphabet $[k]$.

Lemma 2.6. *For all $k \geq 1$,*

$$\frac{d}{dq} W_k(x,q) \mid_{q=1} = \frac{x^{\ell+2}}{(1-kx)^2} \left(2\binom{k}{3} + \binom{k}{2} \right).$$

Proof. We compute the number of ℓ-peaks in all the words of length n over alphabet $[k]$. By differentiating (2.3) with respect to q, we obtain

$$V_k(x) = \frac{d}{dq} W_k(x,q) \mid_{q=1}$$

$$= \frac{(x^{\ell}(1-W_{k-1}(x,1))+V_{k-1}(x))(1-xW_{k-1}(x,1))}{(1-xW_{k-1}(x,1))^2} - \frac{W_{k-1}(x,1)(x^{\ell}(1-x)(1-W_{k-1}(x,1))-xV_{k-1}(x))}{(1-xW_{k-1}(x,1))^2},$$

and using $W_k(x,1)=\frac{1}{1-kx}$ (easy to prove by induction), we obtain

$$\frac{d}{dq}W_k(x,q)\mid_{q=1}=\frac{x^{\ell+2}}{(1-kx)^2}\left(2\binom{k}{3}+\binom{k}{2}\right), \tag{2.5}$$

as claimed. □

By finding the coefficient of x^n in (2.5) we get the following result

Corollary 2.7. *The total number of ℓ-peaks in all the words of length n over alphabet $[k]$ is given by*

$$(n-1-\ell)k^{n-2-\ell}\left(2\binom{k}{3}+\binom{k}{2}\right).$$

We plan to find the explicit formula for the generating function $P_k(x,q)$ for the number of $P_{n,k}$ according to the number of ℓ-peaks.

$$P_k(x,q)=\sum_{n\geq 0}x^n\left(\sum_{\pi\in P_{n,k}}q^{\ell-peak(\pi)}\right).$$

Corollary 2.8. *For all $k\geq 1$, the generating function $P_k(x,q)$ is given by*

$$x^k\prod_{j=1}^{k}\big(W_j(x,q)(1+x^{\ell-1}(x-1))+x^{\ell-1}+qx^{\ell-1}(W_j(x,q)(1-x)-1)\big).$$

Proof. By substituting $q_i=1$ for $i\neq\ell$, and $q_\ell=q$ in Theorem (2.2). □

Lemma 2.9. *For all $k\geq 3$,*

$$\frac{d}{dq}P_k(x,q)\mid_{q=1}=\frac{x^{k+\ell}\binom{k}{2}}{(1-x)\cdots(1-kx)}+\frac{x^{k+\ell+2}}{(1-x)\cdots(1-kx)}\sum_{j=3}^{k}\frac{2\binom{j}{3}+\binom{j}{2}}{(1-jx)}.$$

Proof. By Corollary 2.8, we have

$$\frac{d}{dq}P_k(x,q)\mid_{q=1}=P_k(x,1)\sum_{j=1}^{k}\lim_{q\to 1}\left(\frac{\frac{d}{dq}L_j(q)}{L_j(q)}\right), \tag{2.6}$$

where

$$L_j(q)=(W_j(x,q)(1+x^{\ell-1}(x-1))+x^{\ell-1}+qx^{\ell-1}(W_j(x,q)(1-x)-1).$$

Note that

$$\lim_{q\to 1}\frac{d}{dq}L_j(q)=\lim_{q\to 1}\left(\frac{d}{dq}W_j(x,q)+x^{\ell-1}(W_j(x,q)(1-x)-1)\right)$$
$$=\frac{x^{\ell}\left((j-1)-(j-1)jx+(2\binom{j}{3}+\binom{j}{2})x^2\right)}{(1-jx)^2}=x^{\ell}\left(\frac{j-1}{1-jx}+\frac{(2\binom{j}{3}+\binom{j}{2})x^2}{(1-jx)^2}\right).$$

Hence, by using (2.6) we obtain

$$\frac{d}{dq}P_k(x,q)\mid_{q=1}=\frac{x^{k+\ell}}{(1-x)\cdots(1-kx)}\sum_{j=1}^{k}\left(j-1+\frac{(2\binom{j}{3}+\binom{j}{2})x^2}{(1-jx)}\right)$$
$$=\frac{x^{k+\ell}\binom{k}{2}}{(1-x)\cdots(1-kx)}+\frac{x^{k+\ell+2}}{(1-x)\cdots(1-kx)}\sum_{j=3}^{k}\frac{2\binom{j}{3}+\binom{j}{2}}{(1-jx)},$$

as required. □

By using the facts that $P_k(x,1)=\sum_{n\geq 1}S_{n,k}x^n$ and $\sum_{j=1}^{k}(j-1)x^{\ell}=x^{\ell}\binom{k}{2}$, together with Lemma 2.9 we get the following corollary.

Corollary 2.10. *The total number of the ℓ-peaks in all set partitions $P_{n,k}$ is given by*

$$\binom{k}{2}S_{n-\ell,k}+\sum_{i=\ell+2}^{n-k}S_{n-i,k}\sum_{j=3}^{k}j^{i-\ell-2}\left(2\binom{j}{3}+\binom{j}{2}\right).$$

2.2. Applications

By substituting $\ell=2$ in Corollary 2.7, we obtain the following result

Corollary 2.11. *The total number of the 2-peaks in all the words of length n over alphabet $[k]$ is given by*

$$(n-3)k^{n-4}\left(2\binom{k}{3}+\binom{k}{2}\right).$$

By substituting $\ell=2$ in Corollary 2.10, this leads to

Corollary 2.12. *The total number of the 2-peaks in all set partitions $P_{n,k}$ is given by*

$$\binom{k}{2}S_{n-2,k}+\sum_{i=4}^{n-k}S_{n-i,k}\sum_{j=3}^{k}j^{i-4}\left(2\binom{j}{3}+\binom{j}{2}\right).$$

By substituting $q=0$ in (2.4), we obtain that the generating function for the number of words of length n over alphabet $[k]$ without ℓ-peaks is given by

$$W_k(x,0)=\frac{-x^{\ell}(U_{k-1}(t)-U_{k-2}(t))}{U_k(t)-U_{k-1}(t)-(1+x^{\ell})(U_{k-1}(t)-U_{k-2}(t))},\tag{2.7}$$

where $t = 1 + \frac{x^{\ell+1}}{2}$ and U_m is m-th Chebyshev polynomial of the second kind. By substituting $q = 0$ in Corollary 2.8, we get

$$P_k(x,0) = x^k \prod_{j=1}^{k} \left(W_j(x,0)(1 + x^{\ell-1}(x-1)) + x^{\ell-1} \right), \qquad (2.8)$$

by substituting (2.7) into (2.8), and using the relation $U_{j+1}(t) = 2tU_j(t) - U_{j-1}(t)$, we get

$$P_k(x,0) = x^k \prod_{j=1}^{k} \frac{U_{j-1}(t) - (1+x^\ell)U_{j-2}(t)}{(1-x)U_{j-1}(t) - U_{j-2}(t)},$$

where $t = 1 + \frac{x^{\ell+1}}{2}$, which is the generating function of $P_{n,k}$ without ℓ-peaks. By using the above result with $\ell = 1$, we obtain the same result of Mansour and Shattuck (see [7]).

Corollary 2.13. *The generating function for the number of set partitions of P_n without ℓ-peaks is given by*

$$1 + \sum_{k\geq 1} P_k(x,0) = \sum_{k\geq 0} x^k \prod_{j=1}^{k} \frac{U_{j-1}(t) - (1+x^\ell)U_{j-2}(t)}{(1-x)U_{j-1}(t) - U_{j-2}(t)},$$

where $t = 1 + \frac{x^{\ell+1}}{2}$ and U_m is the m-th Chebyshev polynomial of the second kind.

2.3. Conclusion

In the present paper, we determined the generating function for the number of k-ary words of length n according to the number of ℓ-peaks. Also, we determined the generating function for the number of set partitions of $[n]$ with exactly k blocks according to the number of ℓ-peaks. Seems our techniques can be extended to the case of compositions of n (a composition of n is a word $\sigma_1\sigma_2\cdots\sigma_m$ such that $\sum_{i=1}^{m} \sigma_i = n$), where we leave it to the interest reader.

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