Upper bounds on van der Waerden type numbers for some second order linear recurrence sequences

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Abstract

For suitable integers α, γ and $f : [3, +\infty[\, \cap \, \mathbb{Z} \to [0, +\infty[\, \cap \, \mathbb{Z}$, denote by $w(\mathcal{R}_{\alpha,\gamma,f}, k, r)$ the least positive integer such that for any r-colouring of $[1, w(\mathcal{R}_{\alpha,\gamma,f}, k, r)] \, \cap \, \mathbb{Z}$, there exists a monochromatic finite sequence $(x_1, \ldots, x_k)$ satisfying $x_i = (\alpha a_i + 2)x_{i-1} + (\gamma a_i - 1)x_{i-2}$ with some integers $a_i = 0$ or $a_i \geq f(i)$ $(i = 3, \ldots, k)$. In the present paper we describe the possible values of α and γ. We also derive an upper bound on $w(\mathcal{R}_{\alpha,\gamma,f}, k, 2)$ in these cases. This gives a generalization of a result of B. M. Landman [3].

Keywords: van der Waerden type numbers, linear recurrence sequences

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1. Introduction

Most results of Ramsey theory in the area of number theory deal with monochromatic sequences or monochromatic solutions of diophantine equations, systems of diophantine equations (for an extensive survey see [4]). In this paper we study the monochromatic properties of some second order linear recurrence sequences.

Let $\mathcal{S}$ be a non-empty set of sequences of positive integers. On a finite sequence of $\mathcal{S}$ of length k we mean the first k elements of a sequence from $\mathcal{S}$. For integers $k \geq 3$ and $r \geq 2$, let $w(\mathcal{S}, k, r)$ be the least positive integer if it exists, such that

for any r-colouring of $[1, w(\mathcal{S}, k, r)] \cap \mathbb{Z}$, there is a monochromatic finite sequence of $\mathcal{S}$ of length k. We call $w(\mathcal{S}, k, r)$ a van der Waerden type number.

Throughout this paper by arithmetic progression we mean a strictly increasing arithmetic progression of positive integers and denote their set by $\mathcal{A}$. By the classical theorem of B. L. van der Waerden [6], $w(\mathcal{A}, k, r)$ exists for arbitrary k, r. We will use the standard notation $w(k, r)$ for $w(\mathcal{A}, k, r)$.

Obviously, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are non-empty sets of sequences of positive integers such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $w(\mathcal{S}_1, k, r)$ exists, then $w(\mathcal{S}_2, k, r)$ also exists and $w(\mathcal{S}_2, k, r) \leq w(\mathcal{S}_1, k, r)$. In particular, if $\mathcal{S}$ is a non-empty set of sequences of positive integers with $\mathcal{A} \subseteq \mathcal{S}$, then $w(\mathcal{S}, k, r)$ exists and $w(\mathcal{S}, k, r) \leq w(k, r)$.

In our paper we consider the case of linear recurrence sequences. Remark that we can describe $\mathcal{A}$ by a linear recurrence, namely $\mathcal{A}$ is the set of sequences $(x_i)_{i=1}^{\infty}$ satisfying $x_i = 2x_{i-1} - x_{i-2}$ $(i = 3, 4, \ldots)$ with some positive integers $x_1 < x_2$.

Denote by $\mathcal{F}$ the set of strictly increasing sequences of positive integers satisfying the Fibonacci recurrence, that is

$$\mathcal{F} = \{(x_i)_{i=1}^{\infty} \mid x_1 < x_2 \text{ positive integers}, x_i = x_{i-1} + x_{i-2} \ (i = 3, 4, \ldots)\}.$$

H. Harborth, S. Maasberg [2] and H. Ardal, D. S. Gunderson, V. Jungić, B. M. Landman, K. Williamson [1] proved that $w(\mathcal{F}, k, r)$ exists if and only if $k = 3$. The previous authors also examined other binary recurrences. A forthcoming paper of G. Nyul and B. Rauf [5] studies the existence of van der Waerden type numbers for higher order linear recurrence sequences.

B. M. Landman [3] (see also [4], Section 3.6) considered van der Waerden type numbers for three families of some second order linear recurrence sequences, containing $\mathcal{A}$ as a subset. He gave an upper bound for them when $r = 2$. In [4] at the end of Section 3.6, the authors suggest to investigate some similar families of sequences.

The purpose of our paper is to study this question, but not only for some new separate families. We describe all possible families of sequences and give an upper bound for the corresponding van der Waerden type numbers. As we shall see, the three families and the results of B. M. Landman [3] are special cases of our general ones.

2. Description of our families of sequences

Let $\alpha, \gamma \in \mathbb{Z}$, not both zero, and let $f : [3, +\infty[\cap \mathbb{Z} \to [0, +\infty[\cap \mathbb{Z}$. Denote by $\mathcal{R}_{\alpha, \gamma, f}$ the family of sequences $(x_i)_{i=1}^{\infty}$ with positive integers $x_1 < x_2$, satisfying $x_i = (\alpha a_i + 2)x_{i-1} + (\gamma a_i - 1)x_{i-2}$ for some integers a_i where $a_i = 0$ or $a_i \geq f(i)$ $(i = 3, 4, \ldots)$.

Later on we will also consider the special case when f is identically 0. For this we introduce the notation $\mathcal{R}_{\alpha, \gamma} = \mathcal{R}_{\alpha, \gamma, f}$.

According to the slightly different parametrization given by B. M. Landman [3] for families $\mathcal{R}_{0,1,f}, \mathcal{R}_{1,0,f}, \mathcal{R}_{1,-1,f}$, more generally we could set $\alpha, \beta, \gamma, \delta, A \in \mathbb{Z}, \alpha, \gamma$

not both zero, such that $\alpha A + \beta = 2$, $\gamma A + \delta = -1$ and $g : [3, +\infty[\cap \mathbb{Z} \to [A, +\infty[\cap \mathbb{Z}$ and consider the collection of sequences $(x_i)_{i=1}^\infty$ with positive integers $x_1 < x_2$, satisfying the recurrence $x_i = (\alpha b_i + \beta)x_{i-1} + (\gamma b_i + \delta)x_{i-2}$ where $b_i = A$ or $b_i \geq g(i)$ is an integer $(i = 3, 4, \ldots)$. Note that in fact this is not a more general family of sequences, because it can be reparametrized to $\mathcal{R}_{\alpha, \gamma, f}$ with $g(i) = f(i) + A$ and $b_i = a_i + A$.

The van der Waerden type number $w(\mathcal{R}_{\alpha, \gamma, f}, k, r)$ is meaningful only if each element of $\mathcal{R}_{\alpha, \gamma, f}$ consists of positive integers. But in this case $w(\mathcal{R}_{\alpha, \gamma, f}, k, r)$ always exists, since $\mathcal{A} \subseteq \mathcal{R}_{\alpha, \gamma, f}$ (with the choice $a_i = 0$), moreover $w(\mathcal{R}_{\alpha, \gamma, f}, k, r) \leq w(k, r)$. Thus it is natural to prove the following statement.

Proposition 2.1. *Each element of $\mathcal{R}_{\alpha, \gamma, f}$ contains only positive integers if and only if $\alpha \geq 0$, $\gamma > 0$ or $\alpha > 0$, $\gamma \leq 0$, $\alpha \geq |\gamma|$.*

Proof.
I. First let $\alpha \geq 0$ and $\gamma > 0$. In this case we prove by induction that each element $(x_i)_{i=1}^\infty$ of $\mathcal{R}_{\alpha, \gamma, f}$ is strictly increasing. It follows from the assumption that $x_1 < x_2$. If we suppose $x_{i-1} < x_i$, then $x_{i+1} - x_i = (\alpha a_{i+1} + 1)x_i + (\gamma a_{i+1} - 1)x_{i-1} \geq x_i - x_{i-1} > 0$ since $\alpha a_{i+1} + 1 \geq 1$ and $\gamma a_{i+1} - 1 \geq -1$.

In the case $\alpha > 0$, $\gamma \leq 0$, $\alpha \geq |\gamma|$ we can prove it similarly by induction and using $x_{i+1} - x_i = (\alpha a_{i+1} + 1)x_i + (\gamma a_{i+1} - 1)x_{i-1} \geq (|\gamma| a_{i+1} + 1)(x_i - x_{i-1}) > 0$.

II. In the remaining cases we can find a sequence from $\mathcal{R}_{\alpha, \gamma, f}$ which contains a negative number.

In the case $\alpha < 0$, let $x_1 = 1$. Then we have $x_3 = (\alpha x_2 + \gamma)a_3 + 2x_2 - 1$. If x_2 is sufficiently large, then $\alpha x_2 + \gamma < 0$, hence by choosing a sufficiently large a_3, x_3 is negative.

If $\alpha = 0$ and $\gamma < 0$, we get similarly with the choice $x_1 = 1$ that $x_3 = \gamma a_3 + 2x_2 - 1$, which is negative for sufficiently large a_3.

Finally consider $\alpha > 0$, $\gamma < 0$, $\alpha < |\gamma|$, and let $x_2 = x_1 + 1$. Now $x_3 = ((\alpha + \gamma)x_1 + \alpha)a_3 + x_1 + 2$ holds. If x_1 is sufficiently large, then $(\alpha + \gamma)x_1 + \alpha < 0$, which gives $x_3 < 0$ with a sufficiently large a_3. $\qquad \square$

3. Upper bounds on van der Waerden type numbers

Now we prove our main result, an upper bound on van der Waerden type numbers for $\mathcal{R}_{\alpha, \gamma, f}$ when the number of colours is 2.

Theorem 3.1.
Case 1: *If $\alpha \geq 0$ and $\gamma > 0$, then*

$$w(\mathcal{R}_{\alpha, \gamma, f}, k, 2) \leq w(\mathcal{R}_{\alpha, \gamma, f}, 3, 2) \prod_{j=4}^{k} [(\alpha + \gamma)f(j) + (\alpha + \gamma)j - \alpha - \gamma + 1].$$

Case 2: If $\alpha > 0$, $\gamma \leq 0$ and $\alpha \geq |\gamma|$, then

$$w(\mathcal{R}_{\alpha,\gamma,f}, k, 2) \leq w(\mathcal{R}_{\alpha,\gamma,f}, 3, 2) \prod_{j=4}^{k} (\alpha f(j) + \alpha j - \alpha + 2).$$

Proof. For brevity let us use the notation $C_{\alpha,\gamma,f}(k)$ for the right-hand sides of the inequalities. We prove the theorem by induction on k. It is obvious for $k = 3$. Suppose that it is true for $k - 1$ ($k \geq 4$) and prove it for k.

Let χ be an arbitrary 2-colouring of $[1, C_{\alpha,\gamma,f}(k)] \cap \mathbb{Z}$ with colours red and blue. By the induction hypothesis there exists a $(k - 1)$-term monochromatic finite sequence $(x_1, \ldots, x_{k-1})$ of $\mathcal{R}_{\alpha,\gamma,f}$ under the colouring χ with elements $x_1, \ldots, x_{k-1} \leq C_{\alpha,\gamma,f}(k - 1)$, say it is red.

Let $y_i = [\alpha(f(k) + i - 1) + 2]x_{k-1} + [\gamma(f(k) + i - 1) - 1]x_{k-2}$ ($i = 1, \ldots, k$). In both cases $y_1 < \ldots < y_k$, $y_i > x_{k-1}$ and $y_i \leq [\alpha(f(k) + k - 1) + 2]x_{k-1} + [\gamma(f(k) + k - 1) - 1]x_{k-2}$ using the assumptions on α and γ. In Case 1 the numbers in brackets are positive and $x_{k-2}, x_{k-1} \leq C_{\alpha,\gamma,f}(k - 1)$, hence $y_i \leq [(\alpha + \gamma)f(k) + (\alpha + \gamma)k - \alpha - \gamma + 1]C_{\alpha,\gamma,f}(k - 1) = C_{\alpha,\gamma,f}(k)$. In Case 2 the first number in brackets is positive and the other is negative, which gives similarly $y_i \leq [\alpha(f(k) + k - 1) + 2]x_{k-1} \leq [\alpha(f(k) + k - 1) + 2]C_{\alpha,\gamma,f}(k - 1) = C_{\alpha,\gamma,f}(k)$. This means $y_i \in [1, C_{\alpha,\gamma,f}(k)] \cap \mathbb{Z}$.

Now we have two possibilities: If some y_i ($i = 1, \ldots, k$) is red, then $(x_1, \ldots, x_{k-1}, y_i)$ is a red finite sequence from $\mathcal{R}_{\alpha,\gamma,f}$ of length k having elements in the desired interval. On the other hand, if each y_i ($i = 1, \ldots, k$) is blue, then $(y_1, \ldots, y_k)$ is a k-term monochromatic finite arithmetic progression, hence a finite sequence of $\mathcal{R}_{\alpha,\gamma,f}$ with elements in $[1, C_{\alpha,\gamma,f}(k)] \cap \mathbb{Z}$. □

If f is identically 0, we have the following immediate consequence:

Corollary 3.2.
Case 1: If $\alpha \geq 0$ and $\gamma > 0$, then

$$w(\mathcal{R}_{\alpha,\gamma}, k, 2) \leq \frac{w(\mathcal{R}_{\alpha,\gamma}, 3, 2)}{(\alpha + \gamma + 1)(2\alpha + 2\gamma + 1)} \prod_{j=1}^{k} [(\alpha + \gamma)j - \alpha - \gamma + 1].$$

Case 2: If $\alpha > 0$, $\gamma \leq 0$ and $\alpha \geq |\gamma|$, then

$$w(\mathcal{R}_{\alpha,\gamma}, k, 2) \leq \frac{w(\mathcal{R}_{\alpha,\gamma}, 3, 2)}{2(\alpha + 2)(2\alpha + 2)} \prod_{j=1}^{k} (\alpha j - \alpha + 2).$$

4. Examples

Finally we show some examples with the most interesting possible values of α and γ. Examples 1 and 2 belong to Case 1, while Examples 3, 4 and 5 belong to Case 2.

We notice that Examples 1, 3 and 4 were the original families treated by B. M. Landman [3].

In each example we describe the recurrence, but omit the conditions of $f : [3, +\infty[\, \cap \, \mathbb{Z} \to [0, +\infty[\, \cap \, \mathbb{Z}$, and $a_i = 0$ or $a_i \geq f(i)$, since they are common in all cases. Additionally we give a possible reparametrization of the recurrence, together with the corresponding value of A with our earlier notation. (In Examples 2 and 5, $n!!$ denotes the semifactorial of a natural number n.)

Example 1: $\alpha = 0$, $\gamma = 1$.
Recurrence: $x_i = 2x_{i-1} + (a_i - 1)x_{i-2}$
Reparametrization: $x_i = 2x_{i-1} + b_i x_{i-2}$ $(A = -1)$
Upper bounds:

$$w(\mathcal{R}_{0,1,f}, k, 2) \leq w(\mathcal{R}_{0,1,f}, 3, 2) \prod_{j=4}^{k} (f(j) + j)$$

$$w(\mathcal{R}_{0,1}, k, 2) \leq \frac{7}{6} k!, \text{ since } w(\mathcal{R}_{0,1}, 3, 2) = 7.$$

Example 2: $\alpha = 1$, $\gamma = 1$.
Recurrence: $x_i = (a_i + 2)x_{i-1} + (a_i - 1)x_{i-2}$
Reparametrization: $x_i = (b_i + 3)x_{i-1} + b_i x_{i-2}$ $(A = -1)$
Upper bounds:

$$w(\mathcal{R}_{1,1,f}, k, 2) \leq w(\mathcal{R}_{1,1,f}, 3, 2) \prod_{j=4}^{k} (2f(j) + 2j - 1)$$

$$w(\mathcal{R}_{1,1}, k, 2) \leq \frac{3}{5} (2k - 1)!!, \text{ since } w(\mathcal{R}_{1,1}, 3, 2) = 9.$$

Example 3: $\alpha = 1$, $\gamma = 0$.
Recurrence: $x_i = (a_i + 2)x_{i-1} - x_{i-2}$
Reparametrization: $x_i = b_i x_{i-1} - x_{i-2}$ $(A = 2)$
Upper bounds:

$$w(\mathcal{R}_{1,0,f}, k, 2) \leq w(\mathcal{R}_{1,0,f}, 3, 2) \prod_{j=4}^{k} (f(j) + j + 1)$$

$$w(\mathcal{R}_{1,0}, k, 2) \leq \frac{1}{3} (k + 1)!, \text{ since } w(\mathcal{R}_{1,0}, 3, 2) = 8.$$

Example 4: $\alpha = 1$, $\gamma = -1$.
Recurrence: $x_i = (a_i + 2)x_{i-1} + (-a_i - 1)x_{i-2}$
Reparametrization: $x_i = b_i x_{i-1} + (-b_i + 1)x_{i-2}$ $(A = 2)$
Upper bounds:

$$w(\mathcal{R}_{1,-1,f}, k, 2) \leq w(\mathcal{R}_{1,-1,f}, 3, 2) \prod_{j=4}^{k} (f(j) + j + 1)$$

$$w(\mathcal{R}_{1,-1}, k, 2) \leq \frac{7}{24}(k+1)!, \text{ since } w(\mathcal{R}_{1,-1}, 3, 2) = 7.$$

Example 5: $\alpha = 2$, $\gamma = -1$.
Recurrence: $x_i = (2a_i + 2)x_{i-1} + (-a_i - 1)x_{i-2}$
Reparametrization: $x_i = 2b_i x_{i-1} - b_i x_{i-2}$ $(A = 1)$
Upper bounds:

$$w(\mathcal{R}_{2,-1,f}, k, 2) \leq w(\mathcal{R}_{2,-1,f}, 3, 2) \prod_{j=4}^{k}(2f(j) + 2j)$$

$$w(\mathcal{R}_{2,-1}, k, 2) \leq \frac{3}{16}(2k)!!, \text{ since } w(\mathcal{R}_{2,-1}, 3, 2) = 9.$$

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