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Harmonic sections on the tangent bundle of order two*

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Abstract

The problem studied in this paper is related to the Harmonicity of sections from a Riemannian manifold (M, g) into its tangent bundle of order two T^2M equipped with the Diagonal metric g^D. First we introduce a connection on $\Gamma(T^2M)$ and we investigate the geometry and the harmonicity of sections as maps from (M, g) to (T^2M, g^D).

Keywords: Horizontal lift, vertical lift, harmonic maps.

MSC: 53A45, 53C20, 58E20

1. Introduction

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g \tag{1.1}$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_M h(\tau(\phi), V)\, dv_g, \tag{1.2}$$

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where

$$\tau(\phi) = \text{trace}_g \nabla d\phi \tag{1.3}$$

is the tension field of ϕ. Then we have

Theorem 1.1. *A smooth map $\phi : (M^m, g) \to (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{1.4}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N respectively, then equation (1.4) takes the form

$$\tau(\phi)^\alpha = \left(\Delta\phi^\alpha + g^{ij}\,{}^N\Gamma^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right) = 0, \quad 1 \leq \alpha \leq n, \tag{1.5}$$

where $\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j}\right)$ is the Laplace operator on (M^m, g) and ${}^N\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols on N. One can refer to [1, 4, 6, 7, 8, 9] for background on harmonic maps.

2. Some results on horizontal and vertical lifts

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart

$$(U, x^i)_{i=1\ldots n}$$

on M induces a local chart $(\pi^{-1}(U), x^i, y^j)_{i,j=1,\ldots,n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by:

$$\begin{aligned}
\mathcal{V}_{(x,u)} &= \text{Ker}(d\pi_{(x,u)}) \\
&= \{a^i\frac{\partial}{\partial y^i}|_{(x,u)};\quad a^i \in \mathbb{R}\} \\
\mathcal{H}_{(x,u)} &= \{a^i\frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij}\frac{\partial}{\partial y^k}|_{(x,u)};\quad a^i \in \mathbb{R}\},
\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i\frac{\partial}{\partial y^i} \tag{2.1}$$

$$X^H = X^i\frac{\delta}{\delta x^i} = X^i\{\frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}\} \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$, $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$ and $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})_{i,j=1,\ldots,n}$ a local frame on TM.

Remark 2.1.

1. if $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by
$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$
$$w^v = \{\overline{w}^k + w^i u^j \Gamma^k_{ij}\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

2. if $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ then its vertical and horizontal lifts are defined by
$$u^V = u^i \frac{\partial}{\partial y^i}$$
$$u^H = u^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \}.$$

Proposition 2.2 (see [10])**.** *Let $F \in \mathfrak{T}^1_p(M)$ be a tensor of type (1,p) (respectively, $G \in \mathfrak{T}^0_p(M)$ a tensor of type (0,p)), then there exist a tensor $\gamma(F) \in \mathfrak{T}^1_{p-1}(TM)$ (respectively, $\gamma(G) \in \mathfrak{T}^0_{p-1}(TM)$), localy defined by*

$$\gamma(F) = F^k_{h_1 h_2 \dots h_p} y^{h_1} \frac{\partial}{\partial y^k} \otimes dx^{h_2} \otimes \cdots \otimes dx^{h_p} \tag{2.3}$$

$$\gamma(G) = G_{h_1 h_2 \dots h_p} y^{h_1} dx^{h_2} \otimes \cdots \otimes dx^{h_p} \tag{2.4}$$

where $F = F^j_{i_1 \dots i_p} \frac{\partial}{\partial x^j} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ and $G = G_{i_1 \dots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}$.

Definition 2.3. The Sasaki metric g^s on the tangent bundle TM of M is given by

1. $g^s(X^H, Y^H) = g(X,Y) \circ \pi$
2. $g^s(X^H, Y^V) = 0$
3. $g^s(X^V, Y^V) = g(X,Y) \circ \pi$

for all vector fields $X, Y \in \Gamma(TM)$.

In the general case, Sasaki metrics is considered in [2, 5, 7, 10].

Proposition 2.4 (see [6])**.** *A vector fields $X : (M,g) \to (TM, g^s)$ is harmonic iff*

$$\sum_{i=1} X^k_{ii} = 0, \qquad \sum_{i=1} R^k_{ilj} X^j_i = 0.$$

where X^k_i (resp X^k_{ij}) are the components of the first (resp second) covariant differential of the vector field X.

From Proposition 2.4 we deduce

Proposition 2.5. *If $X : (M, g) \to (TM, g^s)$ is a harmonic vector field, then*

$$\mathrm{trace}_g \nabla^2 X = 0, \qquad \mathrm{trace}_g R(X, \nabla_* X)* = 0.$$

Let M be an n-dimensional manifold. The tangent bundle of order 2 is the natural bundle of 2-jets of differentiable curves, defined by:

$$T^2M = \{j_0^2\gamma \quad ; \quad \gamma : \mathbb{R}_0 \to M, \text{ is a smooth curve at } 0 \in \mathbb{R}\}$$
$$\pi_2 : T^2M \to M$$
$$j_0^2\gamma \mapsto \gamma(0)$$

A local chart $(U, x^i)_{i=1\ldots n}$ on M induces a local chart $(\pi_2^{-1}(U), x^i, y^i, z^i)_{i=1\ldots n}$ on T^2M by the following formulae

$x^i = \gamma^i(0)$.

$y^i = \frac{d}{dt}\gamma^i(0)$.

$z^i = \frac{d^2}{dt^2}\gamma^i(0)$.

Proposition 2.6. *Let M, be an n-dimensional manifold, then TM is sub-bundle of T^2M, and the map*

$$i : TM \to T^2M$$
$$j_0^1 f = j_0^2 \widetilde{f} \tag{2.5}$$

is an injective homomorphism of a natural bundles (not of vector bundles), where

$$\widetilde{f^i} = \int_0^t f^i(s)ds - tf^i(0) + f^i(0) \quad i = 1 \ldots n.$$

Proof. Locally if (U, x^i) is a chart on M and (U, x^i, y^i) and (U, x^i, y^i, z^i) are the induced chart on TM and T^2M respectivelly, then we have $i : (x^i, y^i) \mapsto (x^i, 0, y^i)$, it follows that i is an injective homomorphism. Remains to show that i is well defined.
Let (U, φ) and (V, ψ) are a charts on M, for any vector $j_0^1 f \in TM$, if we denote

$$\widetilde{f}(t) = \varphi^{-1}(\int_0^t \varphi \circ f(s)ds - t\varphi \circ f(0) + \varphi \circ f(0))$$
$$\widehat{f}(t) = \psi^{-1}(\int_0^t \psi \circ f(s)ds - t\psi \circ f(0) + \psi \circ f(0))$$

then we obtain

$$\varphi \circ \widetilde{f}(0) = \varphi \circ f(0)$$

$$
\begin{aligned}
&= \varphi \circ \widehat{f}(0) \\
\frac{d}{dt}(\varphi \circ \widetilde{f})(0) &= 0 \\
&= \frac{d}{dt}(\varphi \circ \widehat{f})(0) \\
\frac{d^2}{dt^2}(\varphi \circ \widetilde{f})(0) &= \frac{d}{dt}(\varphi \circ f)(0) \\
&= \frac{d^2}{dt^2}(\varphi \circ \widehat{f})(0)
\end{aligned}
$$

which proves that $j_0^2 \widetilde{f} = j_0^2 \widehat{f}$.

Theorem 2.7. *Let* (M, g) *be a Riemannian manifold and* ∇ *be the Levi-Civita connection. If* $TM \oplus TM$ *denotes the Whitney sum, then*

$$
\begin{aligned}
S : T^2M &\to TM \oplus TM \\
j_0^2\gamma &\mapsto (\dot{\gamma}(0), (\nabla_{\dot{\gamma}(0)}\dot{\gamma})(0))
\end{aligned} \tag{2.6}
$$

is a diffeomorphism of natural bundles.

In the induced coordinate, we have

$$S : (x^i, y^i, z^i) \mapsto (x^i, y^i, z^i + y^j y^k \Gamma^i_{jk}) \tag{2.7}$$

In the more general case, the difeomorphism S is considered in [3].

Remark 2.8. The diffeomorphism S determines a vector bundle structure on T^2M, for which S be an isomorphism of vector bundles, and $i : TM \to T^2M$ is an injective homomorphism of vector bundles.

Definition 2.9. Let (M, g) be a Riemannian manifold and T^2M be its tangent bundle of order 2 endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$, we define two vector fields on M by:

$$
\begin{aligned}
X_\sigma &= P_1 \circ S \circ \sigma \\
Y_\sigma &= P_2 \circ S \circ \sigma
\end{aligned} \tag{2.8}
$$

where P_1 and P_2 denotes the first and the second projection from $TM \oplus TM$ on TM.

Remark 2.10. We can easily verify that for all sections $\sigma, \varpi \in \Gamma(T^2M)$ and $\alpha \in \mathbb{R}$, we have

$$
\begin{aligned}
X_{\alpha\sigma+\varpi} &= \alpha X_\sigma + X_\varpi \\
Y_{\alpha\sigma+\varpi} &= \alpha Y_\sigma + Y_\varpi
\end{aligned}
$$

From the Remarks 2.8 and 2.10 we can define a connection on $\Gamma(T^2M)$.

Definition 2.11. Let (M,g) be a Riemannian manifold and T^2M be its tangent bundle of order 2 endowed with the vectorial structure induced by the diffeomorphism S. We define a connection on $\Gamma(T^2M)$ by:

$$\begin{aligned}\widehat{\nabla} : \Gamma(TM) \times \Gamma(T^2M) &\to \Gamma(T^2M)\\ (Z,\sigma) &\mapsto \widehat{\nabla}_Z\sigma = S^{-1}((\nabla_Z X_\sigma, \nabla_Z Y_\sigma))\end{aligned} \tag{2.9}$$

where ∇ is the Levi-Civita connection on M.

From formula 2.7 and Definition 2.9 , it follows

Proposition 2.12. *If (U,x^i) is a chart on M and $(\sigma^i,\overline{\sigma}^i)$ are the components of section $\sigma \in \Gamma(T^2M)$ then*

$$\begin{aligned}X_\sigma &= \sigma^i \frac{\partial}{\partial x^i}\\ Y_\sigma &= (\overline{\sigma}^k + \sigma^i\sigma^j\Gamma^k_{ij})\frac{\partial}{\partial x^k}\end{aligned}$$

From Theorem 2.7 and Remark 2.10 we have

Proposition 2.13. *Let (M,g) be a Riemannian manifold and T^2M be its tangent bundle of order 2, then*

$$\begin{aligned}J : \Gamma(TM) &\to \Gamma(T^2M)\\ Z &= S^{-1}(Z,0)\end{aligned} \tag{2.10}$$

is an injective homomorphism of vector bundles.

Locally if (U,x^i) is a chart on M and (U,x^i,y^i) and (U,x^i,y^i,z^i) are the induced chart on TM and T^2M respectivelly, then we have

$$J : (x^i,y^i) \mapsto (x^i,y^i,-y^jy^k\Gamma^i_{jk}) \tag{2.11}$$

Definition 2.14. Let (M,g) be a Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M. For $\lambda = 0,1,2$, the λ-lift of X to T^2M is defined by

$$\begin{aligned}X^0 &= S_*^{-1}(X^H,X^H)\\ X^1 &= S_*^{-1}(X^V,0)\\ X^2 &= S_*^{-1}(0,X^V)\end{aligned} \tag{2.12}$$

In the more general case, the λ-lift is considered in [3].

Theorem 2.15 (see [3]). *Let (M,g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X,Y \in \Gamma(TM)$ and $p \in T^2M$ we have*

1. $[X^0,Y^0]_p = [X,Y]^0_p - (R(X,Y)u)^1 - (R(X,Y)w)^2$

2. $[X^0, Y^i] = (\nabla_X Y)^i$

3. $[X^i, Y^j] = 0.$

where $(u, w) = S(p)$ *and* $i, j = 1, 2$.

Definition 2.16. Let (M, g) be a Riemannian manifold. For any section $\sigma \in \Gamma(T^2M)$ we define the vertical lift of σ to T^2M by

$$\sigma^V = S_*^{-1}(X_\sigma^V, Y_\sigma^V) \in \Gamma(T(T^2M)). \tag{2.13}$$

Remark 2.17. From Definition 2.9 and the formulae (2.5), (2.10), (2.12) and (2.13), for all $\sigma \in \Gamma(T^2M)$ and $Z \in \Gamma(TM)$, we obtain

- $\sigma^V = X_\sigma^1 + Y_\sigma^2$
- $(\widehat{\nabla}_Z \sigma)^V = (\nabla_Z X_\sigma)^1 + (\nabla_Z Y_\sigma)^2$
- $Z^1 = J(Z)^V$
- $Z^2 = i(Z)^V$

3. Metric diagonal and harmonicity

Using Definition 2.3 and formula (2.12), we have

Theorem 3.1. *Let* (M, g) *be a Riemannian manifold and* TM *its tangent bundle equipped with the Sasakian metric* g^s*, then*

$$g^D = S_*^{-1}(\widetilde{g} \oplus \widetilde{g})$$

is the only metric that satisfies the following formulae

$$g^D(X^i, Y^j) = \delta_{ij} \cdot g(X, Y) \circ \pi_2 \tag{3.1}$$

for all vector fields $X, Y \in \Gamma(TM)$ *and* $i, j = 0, \ldots, 2$*, where* $\widetilde{g}$ *is the metric defined by*

$$\begin{aligned}
\widetilde{g}(X^H, Y^H) &= \frac{1}{2} g^s(X^H, Y^H) \\
\widetilde{g}(X^H, Y^V) &= g^s(X^H, Y^V) \\
\widetilde{g}(X^V, Y^V) &= g^s(X^V, Y^V),
\end{aligned}$$

g^D is called the diagonal lift of g to T^2M.

Theorem 3.2. *Let* (M, g) *be a Riemannian manifold and* $\widetilde{\nabla}$ *be the Levi-Civita connection of the tangent bundle of order two* T^2M *equipped with the diagonal metric* g^D*. Then*

1. $(\widetilde{\nabla}_{X^0}Y^0)_p = (\nabla_X Y)^0 - \frac{1}{2}(R(X,Y)u)^1 - \frac{1}{2}(R(X,Y)w)^2$,
2. $(\widetilde{\nabla}_{X^0}Y^1)_p = (\nabla_X Y)^1 + \frac{1}{2}(R(u,Y)X)^0$,
3. $(\widetilde{\nabla}_{X^0}Y^2)_p = (\nabla_X Y)^2 + \frac{1}{2}(R(w,Y)X)^0$,
4. $(\widetilde{\nabla}_{X^1}Y^0)_p = \frac{1}{2}(R_x(u,X)Y))^0$,
5. $(\widetilde{\nabla}_{X^2}Y^0)_p = \frac{1}{2}(R_x(w,X)Y))^0$,
6. $(\widetilde{\nabla}_{X^i}Y^j)_p = 0$

for all vector fields $X, Y \in \Gamma(TM)$ and $p \in \Gamma(T^2M)$, where $i, j = 1, 2$ and $(u, w) = S(p)$.

The proof of theorem 3.2 follows directly from Theorem 3.1 and the Kozul formula.

Lemma 3.3. *Let (M, g) be a Riemannian manifold and (TM, g^s) be the tangent bundle equipped with the Sasaki metric. If $X, Y \in \Gamma(TM)$ are a vector fields and $(x, u) \in TM$ such that $X_x = u$, then we have*

$$d_xX(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, y^j)$ be the induced chart on TM, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x$, then

$$d_xX(Y_x) = Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} + Y^i(x)\frac{\partial X^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)},$$

thus the horizontal part is given by

$$\begin{aligned}(d_xX(Y_x))^h &= Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} - Y^i(x)X^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)} \\ &= Y^H_{(x,X_x)}\end{aligned}$$

and the vertical part is given by

$$\begin{aligned}(d_xX(Y_x))^v &= \{Y^i(x)\frac{\partial X^k}{\partial x^i}(x) + Y^i(x)X^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,X_x)} \\ &= (\nabla_Y X)^V_{(x,X_x)}.\end{aligned}$$

Lemma 3.4. *Let (M, g) be a Riemannian manifold and (T^2M, g^D) be the tangent bundle equipped with the diagonal metric. If $Z \in \Gamma(TM)$ and $\sigma \in \Gamma(T^2M)$, then we have*

$$d_x\sigma(Z_x) = Z^0_p + (\widehat{\nabla}_Z\sigma)^V_p. \tag{3.2}$$

where $p = \sigma(x)$.

Proof. Using Lemma 3.3, we obtain

$$\begin{aligned} d_x\sigma(Z) &= dS^{-1}(dX_\sigma(Z), dY_\sigma(Z))_{S(p)} \\ &= dS^{-1}(Z^h, Z^h)_{S(p)} + dS^{-1}((\nabla_Z X_\sigma)^v, (\nabla_Z Y_\sigma)^v)_{S(p)} \\ &= Z_p^0 + (\widehat{\nabla}_Z\sigma)_p^V. \end{aligned}$$

Lemma 3.5. *Let (M, g) be a Riemannian n-dimensional manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric and let $\sigma \in \Gamma(T^2M)$. Then the energy density associated with σ is*

$$e(\sigma) = \frac{n}{2} + \frac{1}{2}\|\widehat{\nabla}\sigma\|^2.$$

where $\|\widehat{\nabla}\sigma\|^2 = \mathrm{trace}_g g(\nabla X_\sigma, \nabla X_\sigma) + \mathrm{trace}_g g(\nabla Y_\sigma, \nabla Y_\sigma)$.

Proof. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame on M, then

$$e(\sigma) = \frac{1}{2}\sum_{i=1}^{n} g^D(d\sigma(e_i), d\sigma(e_i))$$

Using formula 3.2 and Remark 2.17, we obtain

$$\begin{aligned} e(\sigma) &= \frac{1}{2}\sum_{i=1}^{n} g^D(e_i^0, e_i^0) + \frac{1}{2}\sum_{i=1}^{n} g^D((\widehat{\nabla}_{e_i}\sigma)^V, (\widehat{\nabla}_{e_i}\sigma)^V) \\ &= \frac{n}{2} + \frac{1}{2}\|\widehat{\nabla}\sigma\|^2. \end{aligned}$$

Theorem 3.6. *Let (M, g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. Then the tension field associated with $\sigma \in \Gamma(T^2M)$ is*

$$\tau(\sigma) = (\mathrm{trace}_g\widehat{\nabla}^2\sigma)^V + (\mathrm{trace}_g\{R(X_\sigma, \nabla_* X_\sigma) * + R(Y_\sigma, \nabla_* Y_\sigma) *\})^0. \tag{3.3}$$

Proof. Let $x \in M$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $\nabla_{e_i} e_j = 0$, then

$$\begin{aligned} \tau(\sigma)_x &= \sum_{i=1}^{n} (\nabla_{d\sigma(e_i)} d\sigma(e_i))_{\sigma(x)} \\ &= \sum_{i=1}^{n} \left[\nabla_{e_i^0 + (\nabla_{e_i}\sigma)^V}\left(e_i^0 + (\widehat{\nabla}_{e_i}\sigma)^V\right)\right]_{\sigma(x)} \end{aligned}$$

From Theorem 3.2, we obtain

$$\begin{aligned}\tau(\sigma)_x &= \sum_{i=1}^{n}\Big\{\nabla_{e_i^0}e_i^0 + \nabla_{e_i^0}(\nabla_{e_i}X_\sigma)^1 + \nabla_{e_i^0}(\nabla_{e_i}Y_\sigma)^2 + \nabla_{(\nabla_{e_i}X_\sigma)^1}e_i^0 \\ &\quad + \nabla_{(\nabla_{e_i}Y_\sigma)^2}e_i^0\Big\}_{\sigma(x)} \\ &= \sum_{i=1}^{n}\Big\{(\nabla_{e_i}\nabla_{e_i}X_\sigma)^1_{\sigma(x)} + (\nabla_{e_i}\nabla_{e_i}Y_\sigma)^2_{\sigma(x)} + (R_x(X_\sigma(x),\nabla_{e_i}X_\sigma)e_i)^0 \\ &\quad + (R_x(Y_\sigma(x),\nabla_{e_i}Y_\sigma)e_i)^0\Big\}\end{aligned}$$

Theorem 3.7. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. A section $\sigma : M \to T^2M$ is harmonic if and only the following conditions are verified*

$$\begin{aligned}&\mathrm{trace}_g(\nabla^2 X_\sigma) = 0,\\ &\mathrm{trace}_g(\nabla^2 Y_\sigma) = 0,\\ &\mathrm{trace}_g\{R(X_\sigma,\nabla_* X_\sigma)* + R(Y_\sigma,\nabla_* Y_\sigma)*\} = 0.\end{aligned}$$

From Proposition 2.5 and Theorem 3.7 we obtain

Corollary 3.8. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. If $\sigma : M \to T^2M$ is a section such that X_σ and Y_σ are harmonic vector fields, then σ is harmonic.*

Corollary 3.9. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. If $\sigma : M \to T^2M$ is a section such that X_σ and Y_σ are parallel, then σ is harmonic.*

Theorem 3.10. *Let (M,g) be a Riemannian compact manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. Then $\sigma : M \to T^2M$ is a harmonic section if and only if σ is parallel (i.e $\widehat{\nabla}\sigma = 0$).*

Proof. If σ is parallel, from Corollary 3.9, we deduce that σ is harmonic. Inversely. Let σ_t be a compactly supported variation of σ defined by $\sigma_t = (1+t)\sigma$. From Lemma 3.5 we have

$$e(\sigma_t) = \frac{n}{2} + \frac{(t+1)^2}{2}\|\widehat{\nabla}\sigma\|^2.$$

If σ is a critical point of the energy functional we have :

$$\begin{aligned}0 &= \frac{d}{dt}E(\sigma_t)_{|t=0},\\ &= \int_M \|\widehat{\nabla}\sigma\|^2 dv_{g^D}\end{aligned}$$

Hence $\widehat{\nabla}\sigma = 0$.

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