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Regularity of certain Banach valued stochastic processes

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Abstract

We consider random processes defined on Banach sequence spaces. Then we seek on conditions of $\mathcal{M}$-regularity of bounded linear operators, where $\mathcal{M}$ denotes any of the usual stochastic modes of convergence.

Keywords: Random process on Banach sequence spaces. Stochastic modes of convergence. Locally finite bounded coverings.

MSC: 62L10, 65B99.

1. Introduction

Non deterministic systems derived from applications of probability theory to a wide real life situations give rise to the investigation of stochastic (or random) processes. This setting allows a quote of indeterminacy that reasonably must be considered according to the way the underlying process evolves in time. Among other basic examples, Markov processes concern to possibly dependent random variables, while Poisson processes concern events that occur continously and independent of one another (cf. [7]).

Tests or experiments observed in discrete times amount to sequences of random variables. The problematic of convergence acceleration methods has been studied for many years with broad applications to numerical integration, to informatics, in solving differential equations, etc. (cf. [15, 2]). Sequence transformations and extrapolations were applied in order to accelerate the convergence of sequences in some well known statistical procedures, for instance bootstrap or jacknife (cf. [5, 4]).

The notion of stochastic regularity under the action of linear transformations applied to sequences of random elements in a Banach space was introduced by

H. Lavastre in 1995 (see [6]). His approach was very general, considering sequences $\{X_n\}_{n=1}^{\infty}$ of random variables on a fixed probability space $(\Omega, \mathcal{A}, \mathrm{P})$ with values in a Banach space $(\mathbb{E}, \|\circ\|)$. Any such sequence induces a map

$$X\colon w \to \{X_n\,(w)\}_{n=1}^{\infty}$$

of Ω into the set $\mathcal{S}\,(\mathbb{E})$ of all sequences of elements of $\mathbb{E}$. Let us suppose that $\mathcal{S}\,(\mathbb{E})$ is a normed space and that X is a *generalized random variable*, i.e. $X^{-1}\,(B) \in \mathcal{A}$ if B is any Borel subset of $\mathbb{E}$. Given a linear functional T on $\mathcal{S}\,(\mathbb{E})$ it is natural to ask whether $T(X)\colon w \to T\left[\{X_n\,(w)\}_{n=1}^{\infty}\right]$ is still a generalized random variable. If this is the case, the preservation of stochastic modes of convergence leaded to several notions of *stochastic regularity* of the sequence $\{X_n\}_{n=1}^{\infty}$ under the action of T. From a theoretic point of view, besides its applications the determination of conditions of stochastic regularity has its own interest. For the resolution of this problem for $\mathbb{E} = \mathcal{L}^p\,(\Omega, \mathbb{F}, \mathrm{P})$, where $1 \leqslant p < \infty$ and $\mathbb{F}$ is a Banach space, the reader can see [6, Th. III, 3, p. 480]. Further, stochastic regularity under the action of certain linear transformations defined by some infinite triangular matrices of complex numbers is established in [6, Th. III, 6 and Th. III, 7, p. 482].

The purpose of this article is to initiate an extension of Lavastre's reseach to stochastic processes in other Banach spaces. Nevertheless, we are aware that this goal is easy to state as well as difficult to fulfil. So, we will restrict its generality to the case of bounded linear operators acting on separable Banach sequence spaces. In order to be self-contained in Prop. 2.1 we will show that the set of random variables $X\colon \Omega \to \mathbb{E}$ between a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and a separable Banach space $\mathbb{E}$ admits a complex vector space structure. It is known that if $\mathbb{E}$ is separable and $X\colon \Omega \to \mathbb{E}$ is a random variable then $\|X\|\colon \Omega \to [0, \infty)$ is a random variable (cf. [8]). Prop. 2.2 and Corollary 2.3 will motivate Definition 3.1 in Section 3, giving a precise meaning to random processes defined by a sequence of random variables on a Banach space $\mathbb{E}$. In this section we will analize some concrete examples constructed on an underlying Hilbert space or on a Banach space of continuous functions (see Ex. 3.3 and Ex. 3.4 below). In Section 4 we consider conditions of stochastic regularity of linear bounded operators acting on a Banach sequence space $\mathcal{S}\,(\mathbb{E})$. In particular, we will observe in Remark 3.2 that our approach is more general than the so *called summation process* defined in [6]. In §4.1 we will establish precise conditions of stochastic regularity related to rather general bounded operators, when $\mathbb{E} = \mathbb{C}$ and $\mathcal{S}\,(\mathbb{E})$ is the uniform Banach space of convergent sequences of complex numbers $\mathrm{c}\,(\mathbb{C})$. Finally, in §4.2 we will establish conditions of stochastic regularity of a class of bounded operators for the Banach space $\mathrm{C}\,[0,1]$ and the Banach sequence space $\mathrm{l}^p\,(\mathrm{C}\,[0,1])$, with $1 < p < \infty$.

Besides some posed questions, we believe that possible ways for further investigations will be open. In order of generality, the former will require some knowledge about the structure of bounded linear operators on Banach sequence spaces. Among a huge literature in this topic we only mention [1, 10, 9].

2. Random variables and Banach sequence spaces

Throughout this article $(\Omega, \mathcal{A}, \mathrm{P})$ will be a probability space, $(\mathbb{E}, \|\circ\|)$ will be a separable Banach space and $\mathfrak{X}$ will be a topological space. By $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathfrak{X})$ we will denote the class of *random variables* $X\colon \Omega \to \mathfrak{X}$, i.e. those functions so that $X^{-1}(B) \in \mathcal{A}$ for all sets $B \in \mathfrak{B}(\mathfrak{X})$, where $\mathfrak{B}(\mathfrak{X})$ is the class of Borel subsets of $\mathfrak{X}$. Indeed, $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathfrak{X})$ is really the quotient of all such random variables when we identify those that differ on a set of P-measure zero.

Proposition 2.1. *If the Banach space* $(\mathbb{E}, \|\circ\|)$ *is separable then* $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ *is a complex vector space.*

Proof. Clearly $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ is endowed with a natural complex vector space structure, and it only remains to see that this structure is valid. Let $\{f_n\}_{n=1}^{\infty}$ be a dense sequence of elements of $\mathbb{E}$. Then any open subset $\mathcal{O}$ of $\mathbb{E} \times \mathbb{E}$ can be written as

$$\mathcal{O} = \bigcup_{(n,m,r)\in\mathbb{N}\times\mathbb{N}\times\mathbb{Q}_{>0}:\mathrm{B}_{\infty}((f_n,f_m),r)\subseteq\mathcal{O}} \mathrm{B}_{\infty}\left(\left(f_n, f_m\right), r\right),$$

where for $(n, m, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Q}_{>0}$ is

$$\mathrm{B}_{\infty}\left(\left(f_n, f_m\right), r\right) = \left\{(g, h) \in \mathbb{E} \times \mathbb{E} : \max\left\{\|f_n - g\|, \|f_m - h\|\right\} < r\right\}.$$

So, if $X_1, X_2 \in \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ the set $(X_1, X_2)^{-1}(\mathcal{O})$ is realized as

$$\bigcup_{(n,m,r)\in\mathbb{N}\times\mathbb{N}\times\mathbb{Q}_{>0}:\mathrm{B}_{\infty}((f_n,f_m),r)\subseteq\mathcal{O}} X_1^{-1}\left(\mathrm{B}\left(f_n, r\right)\right) \cap X_2^{-1}\left(\mathrm{B}\left(f_m, r\right)\right),$$

i.e. $(X_1, X_2)^{-1}(\mathcal{O}) \in \mathcal{A}$. Hence $(X_1, X_2) \in \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E} \times \mathbb{E})$. Since $\mathbb{E}$ is a topological vector space the conclusion now follows immediately. $\square$

Proposition 2.2. *Let* $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$.
(i) Let us write

$$\begin{aligned}
\Omega_{\mathbb{E}}^{\infty}\left(\{X_n\}_{n=1}^{\infty}\right) &\triangleq \left\{w \in \Omega : \{X_n(w)\}_{n=1}^{\infty} \in \mathrm{l}^{\infty}(\mathbb{N},\mathbb{E})\right\}, \\
\Omega_{\mathbb{E}}^{\mathrm{c}}\left(\{X_n\}_{n=1}^{\infty}\right) &\triangleq \left\{w \in \Omega : \{X_n(w)\}_{n=1}^{\infty} \in \mathrm{c}(\mathbb{N},\mathbb{E})\right\}, \\
\Omega_{\mathbb{E}}^{\mathrm{c}_0}\left(\{X_n\}_{n=1}^{\infty}\right) &\triangleq \left\{w \in \Omega : \{X_n(w)\}_{n=1}^{\infty} \in \mathrm{c}_0(\mathbb{N},\mathbb{E})\right\}, \\
\Omega_{\mathbb{E}}^{\mathrm{p}}\left(\{X_n\}_{n=1}^{\infty}\right) &\triangleq \left\{w \in \Omega : \{X_n(w)\}_{n=1}^{\infty} \in \mathrm{l}^{\mathrm{p}}(\mathbb{N},\mathbb{E})\right\},
\end{aligned}$$

with $1 \leqslant p < +\infty$. *The above sets are* $\mathcal{A}$*-measurable and*

$$\Omega_{\mathbb{E}}^{\mathrm{p}}\left(\{X_n\}_{n=1}^{\infty}\right) \subseteq \Omega_{\mathbb{E}}^{\mathrm{c}_0}\left(\{X_n\}_{n=1}^{\infty}\right) \subseteq \Omega_{\mathbb{E}}^{\mathrm{c}}\left(\{X_n\}_{n=1}^{\infty}\right) \subseteq \Omega_{\mathbb{E}}^{\infty}\left(\{X_n\}_{n=1}^{\infty}\right). \tag{2.1}$$

(ii) If $X_n \xrightarrow{a.e.} 0$ *then* $\mathrm{P}\left(\Omega_{\mathbb{E}}^{\mathrm{c}_0}\left(\{X_n\}_{n=1}^{\infty}\right)\right) = 1$.

Proof. (i) It suffices to observe that

$$\Omega_{\mathbb{E}}^{\infty}\left(\{X_n\}_{n=1}^{\infty}\right) = \bigcup_{m=1}^{\infty}\bigcap_{p=1}^{\infty}\{\|X_p\| \leqslant m\},$$

$$\Omega_{\mathbb{E}}^{\mathrm{c}}\left(\{X_n\}_{n=1}^{\infty}\right) = \bigcap_{m=1}^{\infty}\bigcup_{p=1}^{\infty}\bigcap_{q\geqslant p, r\geqslant 0}\{\|X_q - X_{q+r}\| \leqslant 1/m\},$$

$$\Omega_{\mathbb{E}}^{\mathrm{c_0}}\left(\{X_n\}_{n=1}^{\infty}\right) = \bigcap_{m=1}^{\infty}\liminf_{q\to\infty}\{\|X_q\| \leqslant 1/m\}.$$

Further,

$$\Omega_{\mathbb{E}}^{\mathrm{p}}\left(\{X_n\}_{n=1}^{\infty}\right) = \left\{w \in \Omega : \sup_{m\in\mathbb{N}}\sum_{n=1}^{m}\|X_n(w)\|^p < +\infty\right\}$$

and $\{\sum_{n=1}^{m}\|X_n\|^p\}_{m\in\mathbb{N}} \subseteq \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{R})$. Thus $\Omega_{\mathbb{E}}^{\mathrm{p}}\left(\{X_n\}_{n=1}^{\infty}\right) \in \mathcal{A}$, because $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{R})$ is an order complete vector space and $\mathcal{A}$ is a σ-algebra. The inclusions (2.1) are trivial.
(ii) It is trivial. $\square$

Corollary 2.3. *Let $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ so that $X_n \xrightarrow{a.e.} 0$. Then there are induced well defined random variables*

$$X^{\mathrm{c_0}}(w) = \{X_n(w)\}_{n=1}^{\infty},\ X^{\mathrm{c}}(w) = \{X_n(w)\}_{n=1}^{\infty},\ X^{\infty}(w) = \{X_n(w)\}_{n=1}^{\infty},$$

where $w \in \Omega$, with values in the Banach spaces $\mathrm{c_0}(\mathbb{N},\mathbb{E})$, $\mathrm{c}(\mathbb{N},\mathbb{E})$ and $\mathrm{l}_{\infty}(\mathbb{N},\mathbb{E})$ respectively.

Remark 2.4. Convergence in probability is not appropiate in general to derive natural randon variables with values in classical Banach sequence spaces. For instance, let $n = k + 2^v$, $0 \leqslant k < 2^v$, $v \in \mathbb{N}_0$, and set $X_n = n\chi_{[k/2^v,(k+1)/2^v]}$. The sequence $\{X_n\}_{n=1}^{\infty}$ of random variables on the Lebesgue measure space $[0, 1]$ converges in probability to zero and $\Omega_{\mathbb{R}}^{\infty}\left(\{X_n\}_{n=1}^{\infty}\right) = \emptyset$.

Remark 2.5. Previously to the main Definition 3.1 of this article, let us remember the usual stochastic modes of convergence:

1. *Convergence in distribution*

   $X_n \xrightarrow{d} X$ if and only if given $B \in \mathfrak{B}(\mathbb{E})$ so that $\mathrm{P}(\{X \in \partial B\}) = 0$ then $\mathrm{P}(\{X_n \in B\}) \to \mathrm{P}(\{X \in B\})$.

2. *Convergence in probability*

   $X_n \xrightarrow{\mathrm{P}} X$ if and only if $\forall\varepsilon > 0$, $\mathrm{P}(\{\|X_n - X\| \geqslant \varepsilon\}) \to 0$.

3. *Almost everywhere convergence*

   $X_n \xrightarrow{a.e.} X$ if and only if $\mathrm{P}(\{X_n \to X\}) = 1$.

4. *Almost complete convergence*

   $X_n \xrightarrow{a.c.} X$ if and only if $\forall \varepsilon > 0$, $\sum_{n=1}^{\infty} \mathrm{P}\left(\{\|X_n - X\| \geqslant \varepsilon\}\right) < +\infty$.

5. *Convergence in the r-th mean*

   $X_n \xrightarrow{\mathrm{L}^r} X$ if and only if $\mathrm{E}\left(\|X_n - X\|^r\right) \to 0$.

6. *Convergence in the mean*

   $X_n \xrightarrow{\mathrm{E}} X$ if and only if $\mathrm{E}\left(X_n - X\right) \to 0$. (See Remark 2.6 below).

It is well known that almost complete convergence implies almost everywhere convergence, almost everywhere convergence implies convergence in probability and convergence in probability implies convergence in distribution (cf. [12, pp. 240]). Likewise, if $r > s$ then convergence in the r-th mean implies convergence is the s-th mean and the later implies convergence in probability. Further, by Lévy‘s convergence theorem if $X_n \xrightarrow{a.e.} X$ in $\mathcal{M}_{\mathrm{P}}\left(\Omega, \mathcal{A}, \mathbb{R}\right)$ and there is a random variable Y so that for all $n \in \mathbb{N}$ is $|X_n| \leqslant Y$ and $\mathrm{E}\left(Y\right) < +\infty$ then $X_n \xrightarrow{\mathrm{L}^r} X$ (see [14, pp. 187–188]).

Remark 2.6. If the Banach space $\mathbb{E}$ is separable the notion of *expected valu*e of a random variable $X \in \mathcal{M}_{\mathrm{P}}\left(\Omega, \mathcal{A}, \mathbb{E}\right)$ is well defined. Precisely, given a random variable X its expected value is any element $f \in \mathbb{E}$ so that if $\varphi \in \mathbb{E}^*$ then

$$\langle f, \varphi \rangle = \int_{\Omega} \langle X\left(w\right), \varphi \rangle \, d\,\mathrm{P}\left(w\right).$$

Since $\mathbb{E}^*$ becomes a separating family if such an element exists it is necessarily unique and it is denoting as $\mathrm{E}\left(X\right)$. For instance, $\mathrm{E}\left(X\right)$ exists if $\mathrm{E}\left(\|X\|\right) < +\infty$. For further information the reader can see [11].

3. Random processes on Banach sequence spaces

Definition 3.1. A random process of $\mathcal{M}_{\mathrm{P}}\left(\Omega, \mathcal{A}, \mathbb{E}\right)$ on a Banach sequence space $\mathcal{S}\left(\mathbb{E}\right)$ is a sequence $\{X_n\}_{n=1}^{\infty} \cup \{X\} \subseteq \mathcal{M}_{\mathrm{P}}\left(\Omega, \mathcal{A}, \mathbb{E}\right)$ so that:
(i) the set

$$\Omega^{\mathcal{S}(\mathbb{E})}\left(\{X_n - X\}_{n=1}^{\infty}\right)) \triangleq \{w \in \Omega : \{X_n\left(w\right) - X\left(w\right)\}_{n=1}^{\infty} \in \mathcal{S}\left(\mathbb{E}\right)\}$$

belongs to $\mathcal{A}$;
(ii) $\mathrm{P}\left(\Omega^{\mathcal{S}(\mathbb{E})}\left(\{X_n - X\}_{n=1}^{\infty}\right)\right) = 1$. By $\left[\mathcal{M}_{\mathrm{P}}\left(\Omega, \mathcal{A}, \mathbb{E}\right), \mathcal{S}\left(\mathbb{E}\right)\right]$ we will denote the class of all such random processes.

Remark 3.2. By Prop. 2.2 any almost everywhere convergent sequence of random variables with values in a Banach space $\mathbb{E}$ defines a random process on the classical Banach sequence spaces $\mathrm{c}_0\left(\mathbb{N},\mathbb{E}\right)$, $\mathrm{c}\left(\mathbb{N},\mathbb{E}\right)$ and $\mathrm{l}^{\infty}\left(\mathbb{N},\mathbb{E}\right)$.

Example 3.3. Let $1 \leqslant p < \infty$, $T \in \mathcal{B}(\mathcal{L}^p[0,1])$. If $n \in \mathbb{N}$ let $X_n(t) = T^n\left(\chi_{[0,t]}\right)$, $0 \leqslant t \leqslant 1$. If $0 \leqslant s, t \leqslant 1$ then

$$\begin{aligned}\|X_n(t) - X_n(s)\|_p &= \left\|T^n\left(\chi_{[0,t]} - \chi_{[0,s]}\right)\right\|_p \\ &\leqslant \|T^n\| \left\|\chi_{[0,t]\triangle[0,s]}\right\|_p \\ &\leqslant \|T\|^n |s-t|^{1/p},\end{aligned}$$

i.e. $X_n\colon [0,1] \to \mathcal{L}^p[0,1]$ becomes uniformly continuous and

$$\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_{dx}\left([0,1], \mathfrak{L}[0,1], \mathcal{L}^p[0,1]\right),$$

where dx is the Lebesgue measure on $[0,1]$ and $\mathfrak{L}[0,1]$ is the Lebesgue $\sigma-$algebra of subsets of $[0,1]$. For instance, let $Tf(t) = \int_0^t f dx$ if $f \in \mathcal{L}^p[0,1]$. It is easy to see that T is a bounded linear operator and if $n \in \mathbb{N}$ and $0 \leqslant t, \tau \leqslant 1$ then

$$X_n(t)(\tau) \triangleq T^n\left(\chi_{[0,t]}\right)(\tau) = \begin{cases} \left(\tau^n - (\tau-t)^n\right)/n! & \text{if } 0 \leqslant t \leqslant \tau, \\ \tau^n/n! & \text{if } \tau \leqslant t \leqslant 1. \end{cases} \tag{3.1}$$

Consequently, if $t \in [0,1]$ and $n \in \mathbb{N}$ the following inequality

$$\|X_n(t)\|_p \leqslant 1/\left[n!\,(1+np)^{1/p}\right] \tag{3.2}$$

holds. From (3.2) we infer that $X_n \xrightarrow{a.c.} 0$ and that $\{X_n\}_{n=1}^{\infty}$ defines well random process on any of the classical Banach sequence spaces on $\mathcal{L}^p[0,1]$. Further, if $n \in \mathbb{N}$ from (3.1) we have that $X_n\colon [0,1] \to \mathrm{C}[0,1]$ and

$$\|X_n(s) - X_n(t)\|_{\infty} = \max\left\{|s-t|^n, |(1-t)^n - (1-s)^n|\right\}/n!$$

if $0 \leqslant s, t \leqslant 1$, i.e. X_n is continuous and $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_{dx}\left([0,1], \mathfrak{L}[0,1], \mathrm{C}[0,1]\right)$. Since

$$\|X_n(t)\|_{\infty} = \left(1 - (1-t)^n\right)/n!$$

the same conclusions are true for the underlying Banach space $\mathrm{C}[0,1]$. In this setting the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges to zero in the r-th mean for all $r \in \mathbb{N}$. For, if $n \in \mathbb{N}$ and $s \in \mathbb{R}$ we have

$$F_n(s) \triangleq \int_{\{\|X_n\|_{\infty} \leqslant s\}} dx = \begin{cases} 0 & \text{if } s \leqslant 0, \\ 1 - (1-sn!)^{1/n} & \text{if } 0 < s < 1/n!, \\ 1 & \text{if } s \geqslant 1/n!. \end{cases} \tag{3.3}$$

In particular, $\mathrm{d}-\lim_{n\to\infty} \|X_n\|_{\infty} = \mathrm{H}$, i.e. the sequence of random variables $\{\|X_n\|_{\infty}\}_{n=1}^{\infty}$ converges in distribution to the Heaviside function. Now, using (3.3) we obtain

$$\mathrm{E}\left(\|X_n\|_{\infty}^r\right) = \int_0^{1/n!} s^r dF_n(s)$$

$$
\begin{aligned}
&= (n-1)! \int_0^{1/n!} s^r \left(1 - sn!\right)^{1/n-1} ds \\
&= \frac{1}{nn!^r} \int_0^1 u^r \left(1-u\right)^{1/n-1} du \\
&= \frac{1}{nn!^r} \cdot \mathrm{Be}\left(r+1, 1/n\right) \\
&= \frac{1}{nn!^r} \cdot \frac{\Gamma\left(r+1\right)\Gamma\left(1/n\right)}{\Gamma(r+1+1/n)} \\
&= \frac{r!}{nn!^r} \cdot \prod_{j=0}^{r} \left(1/n + j\right)^{-1} \leqslant \frac{r!}{(n-1)!^r},
\end{aligned}
$$

i.e. $\lim_{n\to\infty} \mathrm{E}\left(\|X_n\|_\infty^r\right) = 0$. Further, if $n \in \mathbb{N}$ then

$$
\mathrm{E}\left(X_n\right)(\tau) = \frac{\tau^n}{n!} - \frac{\tau^{n+1}}{(n+1)!}. \tag{3.4}
$$

For, let $\phi \in \mathrm{BV}\left[0,1\right]$ be a complex valued function of bounded variation on $[0,1]$. By the Fubini-Tonelli theorem and (3.1) we see that

$$
\begin{aligned}
\iint\limits_{[0,1]\times[0,1]} \left|X_n\left(t\right)\left(\tau\right)\right| d\left|\phi\right|\left(\tau\right) \times dt &= \int_0^1 \int_0^1 \left|X_n\left(t\right)\left(\tau\right)\right| d\left|\phi\right|\left(\tau\right) dt \\
&\leqslant \int_0^1 d\left|\phi\right|\left(\tau\right)/n! \leqslant \|\phi\|_{\mathrm{BV}[0,1]} < +\infty,
\end{aligned}
$$

where $\|\phi\|_{\mathrm{BV}[0,1]} \triangleq \left|\phi\left(0\right)\right| + \mathrm{V}_{[0,1]}\left(\phi\right)$. As it is well known $\left(\mathrm{BV}\left[0,1\right], \|\circ\|_{\mathrm{BV}[0,1]}\right)$ becomes a Banach space isometrically isomorphic to $\left(\mathrm{C}\left[0,1\right]\right)^*$ (cf. [3, Th. 1.37, p. 16]). Hence,

$$
\begin{aligned}
\left\langle \frac{\tau^n}{n!} - \frac{\tau^{n+1}}{(n+1)!}, d\phi\left(\tau\right) \right\rangle &= \int_0^1 \left(\frac{\tau^n}{n!} - \frac{\tau^{n+1}}{(n+1)!}\right) d\phi\left(\tau\right) \\
&= \int_0^1 \left(\int_0^\tau \frac{\tau^n - \left(\tau - t\right)^n}{n!} dt + \frac{\tau^n}{n!}\left(1-\tau\right)\right) d\phi\left(\tau\right) \\
&= \int_0^1 \int_0^1 X_n\left(t\right)\left(\tau\right) dt d\phi\left(\tau\right) \\
&= \int_0^1 \int_0^1 X_n\left(t\right)\left(\tau\right) d\phi\left(\tau\right) dt \\
&= \int_0^1 \langle X_n(t), d\phi\rangle\, dt.
\end{aligned}
$$

By the uniqueness of the expected value of X_n as it was pointed in Remark 2.6 we obtain (3.4). In particular, $\mathrm{E}\left(X_n\right) \to 0$ in $\mathrm{C}\left[0,1\right]$.

Example 3.4. Let $\Omega = \{00, 010, 0110, \dots\} \cup \{11, 101, 1001, \dots\}$ and if $0 < p < 1$ let $q = 1 - p$. Given $w \in \Omega$ we put $\mathrm{P}(w) = p^a q^b$ if w contains a zeros and b ones. Hence (Ω, P) becomes a discrete probability space. For instance, Ω can be seen as the set of all possible random events in a game consisting in throwing a possible non calibrated coin successively, assuming that the play ends when the first result occurs again. Let us consider a separable Hilbert space $\mathcal{H}$ endowed with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. We can represent any element $w \in \Omega$ as a sequence $w = \{w_m\}_{m=1}^{\infty}$, where $w_m = 0$ except a possible finite number of indices. For instance, we write $010 = \{0, 1, 0, 0, 0, \dots\}$, $1001 = \{1, 0, 0, 1, 0, 0, 0, \dots\}$, etc. Now, for $w \in \Omega$ and $n \in \mathbb{N}$ we will write $Y_n(w) = \sum_{m=1}^{n} w_m \cdot e_m$. Then $\{Y_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{P}(\Omega), \mathcal{H})$. Further, if for $w \in \Omega$ we set

$$Y(w) = \sum_{m=1}^{\infty} w_m \cdot e_m \tag{3.5}$$

then $Y \colon \Omega \to \mathcal{H}$ is a well defined random variable since any series in (3.5) is reduced to a finite sum. If $X_n \triangleq Y_n - Y$, $n \in \mathbb{N}$, clearly $\Omega_{\mathcal{H}}^{\mathrm{P}}\left(\{X_n\}_{n \in \mathbb{N}}\right) = \Omega$. Indeed, $\{X_n\}_{n=1}^{\infty}$ converges to zero in the r-th mean for all $r \in \mathbb{N}$. For, if $n \in \mathbb{N}$ then

$$\begin{aligned}
\mathrm{P}(\{\|X_n\| = 0\}) &= \mathrm{P}\left(\left\{00, 010, \dots, 01\dots1\overset{(n+1)}{0}, 11, 101, \dots, 10\dots0\overset{(n)}{1}\right\}\right) \qquad (3.6)\\
&= p^2 \sum_{j=0}^{n-1} q^j + q^2 \sum_{j=0}^{n-2} p^j \\
&= 1 - pq^n - p^{n-1}q, \\
\mathrm{P}(\{\|X_n\| = 1\}) &= \mathrm{P}\left(\left\{01\dots\overset{(n)}{1}10, 10\dots\overset{(n)}{0}1, 10\dots\overset{(n)}{0}01, \dots\right\}\right) \\
&= p^2 q^n + p^{n-1}q^2 + p^n q^2 + \dots \\
&= p^2 q^n + p^{n-1}q.
\end{aligned}$$

For an integer $m \geqslant 2$ we see that

$$\mathrm{P}\left(\left\{\|X_n\| = m^{1/2}\right\}\right) = \mathrm{P}\left(\left\{01\dots\overset{(n)}{1}1\dots\overset{(n+m)}{1}\right\}\right) = p^2 q^{n+m-1}. \tag{3.7}$$

Using the identities (3.6) and (3.7) we evaluate

$$\begin{aligned}
\mathrm{E}(\|X_n\|^r) &= \sum_{m=0}^{\infty} m^{r/2}\, \mathrm{P}\left(\left\{\|X_n\| = m^{1/2}\right\}\right) \qquad (3.8)\\
&= p^2 q^n + p^{n-1}q + p^2 q^{n-1} \sum_{m=2}^{\infty} m^{r/2} q^m.
\end{aligned}$$

Letting $n \to \infty$ in (3.8) the claim follows. With the notation of Ex. 3.4 we will show that

$$\lim_{n\to\infty} \mathrm{E}\left(X_n\right) = 0. \tag{3.9}$$

For, we will prove that if $n \in \mathbb{N}$ then

$$\mathrm{E}\left(X_n\right) = -\sum_{\upsilon=n+1}^{\infty} \left(pq^{\upsilon-1} + p^{\upsilon-2}q^2\right) e_\upsilon \tag{3.10}$$

and later (3.9) will follows at once. As $0 < p, q < 1$ the above series is absolutely convergent. If $g \in \mathcal{H}$ the random variable $w \to \langle X_n(w), g\rangle$ maps Ω onto the set $\left\{\sum_{s=1}^{k} \langle g, e_{n+s}\rangle\right\}_{k=1}^{\infty}$. If $m \in \mathbb{N}$ set $\Omega_m = \{w \in \Omega : w_\upsilon = 0 \text{ if } \upsilon > m\}$. Thus $\{\Omega_m\}_{m=1}^{\infty}$ is an increasing sequence of sets and $\Omega = \cup\Omega_m$. If $m \in \mathbb{N}$ and $m > n$ we have

$$\begin{aligned}
\int_\Omega \langle X_n(w), g\rangle \chi_{\Omega_m}(w)\, d\,\mathrm{P}(w) &= -\int_{\Omega_m} \sum_{\upsilon=n+1}^{m} w_\upsilon \langle e_\upsilon, g\rangle\, d\,\mathrm{P}(w) \\
&= -\sum_{s=1}^{m} \left\langle \sum_{t=1}^{s} e_{n+t}, g \right\rangle p^2 q^{n+s-1} \\
&\quad - \sum_{\upsilon=n+1}^{m} \langle e_\upsilon, g\rangle p^{\upsilon-2} q^2 \\
&= -p \sum_{t=1}^{m} \langle e_{n+t}, g\rangle \left(q^{n+t-1} - q^{n+m}\right) \\
&\quad - \sum_{\upsilon=n+1}^{m} \langle e_\upsilon, g\rangle p^{\upsilon-2} q^2.
\end{aligned} \tag{3.11}$$

Since the series $\sum_{m=1}^{\infty} q^m m^{1/2}$ converges we conclude that

$$0 \leqslant \limsup_{m\to\infty} q^{n+m} \sum_{t=1}^{m} |\langle e_{n+t}, g\rangle| \leqslant \limsup_{m\to\infty} q^{n+m} \|g\|\, m^{1/2} = 0. \tag{3.12}$$

From (3.11) and (3.12) we get

$$\begin{aligned}
\lim_{m\to\infty} \int_\Omega \langle X_n(w), g\rangle \chi_{\Omega_m}(w)\, d\,\mathrm{P}(w) &= -p \sum_{t=1}^{\infty} \langle e_{n+t}, g\rangle q^{n+t-1} \\
&\quad - \sum_{\upsilon=n+1}^{\infty} \langle e_\upsilon, g\rangle p^{\upsilon-2} q^2 \\
&= -\sum_{\upsilon=n+1}^{\infty} \langle e_\upsilon, g\rangle \left(pq^{\upsilon-1} + p^{\upsilon-2} q^2\right)
\end{aligned} \tag{3.13}$$

$$= \left\langle -\sum_{\upsilon=n+1}^{\infty} \left(pq^{\upsilon-1} + p^{\upsilon-2}q^2\right) e_\upsilon, g \right\rangle.$$

But for $m \in \mathbb{N}$ and $w \in \Omega$ we see that

$$\left|\left\langle X_n(w), g\right\rangle\right| \chi_{\Omega_m}(w) \leqslant \left|\left\langle X_n(w), g\right\rangle\right|. \tag{3.14}$$

Moreover,

$$\begin{aligned} \int_\Omega \left|\left\langle X_n(w), g\right\rangle\right| d\,\mathrm{P}(w) &= \left|\left\langle g, e_{n+1}\right\rangle\right| \mathrm{P}\left(\left\{10\ldots0\overset{(n+1)}{1}, 01\ldots\overset{(n+1)}{1}0\right\}\right) \qquad (3.15) \\ &\quad + \sum_{k=2}^{\infty}\left|\sum_{s=1}^{k}\left\langle g, e_{n+s}\right\rangle\right| \mathrm{P}\left(\left\{01\ldots\overset{(n+1)}{1}\ldots\overset{(n+k)}{1}0\right\}\right) \\ &= \left|\left\langle g, e_{n+1}\right\rangle\right|\left(p^{n-1}q^2 + pq^n\right) \\ &\quad + \sum_{k=2}^{\infty}\left|\sum_{s=1}^{k}\left\langle g, e_{n+s}\right\rangle\right| pq^{n+k-1}. \end{aligned}$$

Further,

$$\sum_{k=1}^{\infty}\left|\sum_{s=1}^{k}\left\langle g, e_{n+s}\right\rangle\right| q^k \leqslant \|g\| \sum_{k=1}^{\infty} k^{1/2} q^k < +\infty. \tag{3.16}$$

Thus, by (3.15) and (3.16) the random variable $w \to \left\langle X_n(w), g\right\rangle$ becomes absolutely integrable on Ω. Finally, using (3.14) and the Lebesgue dominated convergence theorem in (3.13) we obtain

$$\int_\Omega \left\langle X(w), g\right\rangle d\,\mathrm{P}(w) = \left\langle -\sum_{\upsilon=n+1}^{\infty} \left(pq^{\upsilon-1} + p^{\upsilon-2}q^2\right) e_\upsilon, g \right\rangle$$

and (3.10) follows.

4. Random processes and stochastic regularity

Definition 4.1. With the notation of Definition 3.1, let $A \in \mathcal{B}[\mathcal{S}(\mathbb{E})]$. Then A is called $\mathcal{M}$-regular for $\{X_n - X\}_{n=1}^{\infty}$ on the Banach sequence space $\mathcal{S}(\mathbb{E})$ if it preserves its $\mathcal{M}$-stochastic mode of convergence, i.e. if $\mathcal{M}\text{-}\lim_{n\to\infty} X_n = X$ then $\mathcal{M}\text{-}\lim_{m\to\infty} \|A_m(\{X_n - X\}_{n=1}^{\infty})\| = 0$. A subset $\mathcal{R}$ of $\mathcal{S}(\mathbb{E})$ is called $\mathcal{M}$-regular for the sequence $\{X_n - X\}_{n=1}^{\infty}$ on $\mathcal{S}(\mathbb{E})$ if each element of $\mathcal{R}$ is $\mathcal{M}$-regular for it. Indeed, $\mathcal{R}$ will be called simply $\mathcal{M}$-regular on $\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ and $\mathcal{S}(\mathbb{E})$ if each element of $\mathcal{R}$ preserves the $\mathcal{M}$-stochastic mode of convergence of any random process of $[\mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E}), \mathcal{S}(\mathbb{E})]$.

Remark 4.2. The well known shift operator $W((f_n)_{n=1}^{\infty}) = (f_{n+1})_{n=1}^{\infty}$ is linear and bounded on any of the classical Banach sequences spaces $\mathrm{l}^p(\mathbb{N},\mathbb{C})$, $\mathrm{c}_0(\mathbb{N},\mathbb{C})$,

$c(\mathbb{N},\mathbb{C})$ and $l^{\infty}(\mathbb{N},\mathbb{C})$. For conditions concerning to the $\mathcal{M}$-regularity of $p(W)$ when p is any polynomial the reader can see [6]. That approach could be improved in various directions, for instance: (1st) What can be said about the $\mathcal{M}$-regularity of general bounded operators on Banach sequence spaces over $\mathbb{C}$? (2nd) What happens if we state the same problem replacing $\mathbb{C}$ by any other Banach space? The first question already has its own interest since Banach sequence spaces of complex or real numbers offer a natural frame to modeling a huge variety of statistical and numerical analysis processes. Even in this case the determination of the structure and characterization of bounded operators sometimes constitute a difficult matter. In particular, the characterization of bounded operators on $c(\mathbb{N},\mathbb{C})$ is a celebrated result of I. Schur (cf. [13]). For more information on these topics the reader can see [9], [10]. For a proof of Schur‘s theorem and the characterization of bounded operators on Banach sequence spaces of complex series see [1].

4.1. $\mathcal{M}$-regularity on $[\mathcal{M}_{\mathrm{P}}(\Omega,\mathcal{A},\mathbb{C}),c(\mathbb{N},\mathbb{C})]$

If $A\in\mathcal{B}(c(\mathbb{N},\mathbb{C}))$ there is a unique complex matrix $\{a_{n,m}\}_{n,m=0}^{\infty}$ so that for $z\in c(\mathbb{N},\mathbb{C})$ we have

$$A(z)=\left\{a_{n,0}\lambda(z)+\sum_{m=1}^{\infty}a_{n,m}\cdot z_m\right\}_{n=1}^{\infty},$$

where $\lambda(z)=\lim_{n\to\infty}z_n$. Further,

$$\begin{aligned}\|A\|&=\sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}|a_{n,m}|, &(4.1)\\ a_{0,0}&=\lim_{n\to\infty}\sum_{m=1}^{\infty}a_{n,m},\\ a_{0,m}&=\lim_{n\to\infty}a_{n,m}\text{ if }m\in\mathbb{N}\end{aligned}$$

and $\{a_{0,m}\}_{m=1}^{\infty}\in l^{1}(\mathbb{N},\mathbb{C})$ (cf. [1], Corollary 2, p. 20). Let us consider the random process on $c(\mathbb{N},\mathbb{C})$ induced by $X_n=\chi_{[n,+\infty)}$, $n\in\mathbb{N}$ on the probability space $(\mathbb{R},\mathcal{L}(\mathbb{R}),\mathrm{P})$, where $\mathcal{L}(\mathbb{R})$ is the class of Lebesgue measurable subsets of $\mathbb{R}$ and $\mathrm{P}(E)=\int_{E\cap(0,+\infty)}\exp(-x)\,dx$ if $E\in\mathcal{L}(\mathbb{R})$. Let $A\in\mathcal{B}(c(\mathbb{N},\mathbb{C}))$ be defined by the infinite matrix whose nm-entry is

$$a_{n,m}=\begin{cases}1 & \text{if } n=m=0,\\ 0 & \text{if } n=0, m\in\mathbb{N},\\ (1+n)^{-m} & \text{if } n,m\in\mathbb{N}.\end{cases}$$

Then A is ac-regular for the sequence $\{X_n\}_{n=1}^{\infty}$. For, let $\varepsilon>0$, $m\in\mathbb{N}$. Then

$$\sum_{n=1}^{m}\mathrm{P}(\{|X_n|\geqslant\varepsilon\})=\sum_{n=1}^{m}\int_{n}^{+\infty}\exp(-x)dx=\sum_{n=1}^{m}\exp(-n)$$

i.e. $\sum_{n=1}^{\infty} \mathrm{P}\left(\{|X_n| \geqslant \varepsilon\}\right) = 1/\left(\mathrm{e}-1\right)$ and $X_n \xrightarrow{a.c.} 0$. If $A\left(\{X_n\}_{n=1}^{\infty}\right) = \{Y_n\}_{n=1}^{\infty}$ then

$$Y_n = \sum_{m=1}^{\infty} \left(1+n\right)^{-m} \chi_{[m,+\infty)} \text{ if } n \in \mathbb{N}. \tag{4.2}$$

Consequently, for $n \in \mathbb{N}$ and $w \in \mathbb{R}$ it is easy to see that

$$Y_n\left(w\right) = \frac{1}{n}\left(1 - \frac{1}{\left(1+n\right)^{[w]}}\right)\chi_{[0,+\infty)}\left(w\right).$$

Thus $\{|Y_n| \geqslant \varepsilon\} = \varnothing$ if $n > 1/\varepsilon$ and so ac-$\lim_{n\to\infty} Y_n = 0$. However, $\mathrm{c}\left(\mathbb{N},\mathbb{C}\right)$ is not ac-regular for $\{X_n\}_{n=1}^{\infty}$. For, if $B \in \mathcal{B}\left(\mathrm{c}\left(\mathbb{N},\mathbb{C}\right)\right)$ is defined by the infinite matrix whose nm-entry is 2^{-m-1} we write $B\left(\{X_n\}_{n=1}^{\infty}\right) = \{Z_n\}_{n=1}^{\infty}$. For $w \in \mathbb{R}$ we now evaluate that $Z_n(w) = \left(1 - 2^{-[w]}\right)/2$ for all $n \in \mathbb{N}$. If $0 < \varepsilon < 1/2$ let us choose $v \in \mathbb{N}$ so that $\varepsilon < \left(1 - 2^{-v}\right)/2$. Then,

$$\{|Z_n| \geqslant \varepsilon\} \supseteq \left\{Z_n \geqslant 2^{-1} - 2^{-v-1}\right\} = [v, +\infty),$$

i.e. $\mathrm{P}\left(\{|Z_n| \geqslant \varepsilon\}\right) \geqslant \exp\left(-v\right)$. Therefore ac-$\lim_{n\to\infty} |Z_n| \neq 0$ and B is not ac-regular for the sequence $\{X_n\}_{n=1}^{\infty}$. Since obviously B is not a d-regular operator for $\{X_n\}_{n=1}^{\infty}$ it is also not p-regular nor not ae-regular for it. Finally, if $r > 0$ then A becomes L^r-regular for $\{X_n\}_{n=1}^{\infty}$. For,

$$\mathrm{L}^r\text{-}\lim_{n\to\infty} X_n = \lim_{n\to\infty} \mathrm{E}\left(|X_n|^r\right) = \lim_{n\to\infty} \exp\left(-m\right) = 0.$$

If $n \in \mathbb{N}$ using (4.2) Y_n becomes a discrete random variable and

$$\begin{aligned} \mathrm{E}\left(|Y_n|^r\right) &= \frac{1}{n^r}\sum_{m=1}^{\infty}\left(1 - \frac{1}{\left(n+1\right)^m}\right)^r \mathrm{P}\left([m-1, m)\right) \\ &\leqslant \frac{1}{n^r}\sum_{m=1}^{\infty}\left[\exp\left(-m\right) - \exp\left(-m-1\right)\right] = \frac{1}{\mathrm{e}\, n^r}, \end{aligned}$$

i.e. L^r-$\lim_{n\to\infty} Y_n = 0$. However, it is evident that B is not L^r-regular for $\{X_n\}_{n=1}^{\infty}$.

Problem 4.3. Is it possible to characterize the subclasses of $\mathcal{M}$-regular operators of $\mathcal{B}\left(\mathrm{c}\left(\mathbb{N},\mathbb{C}\right)\right)$ for the sequence $\{X_n\}_{n=1}^{\infty}$? In the general case, what relevant properties can be developed concerning to those classes? Can be determinated some subsets of $\mathcal{B}\left(\mathrm{c}\left(\mathbb{N},\mathbb{C}\right)\right)$ that are $\mathcal{M}$-regular for all random process on any unrestricted probability space $(\Omega, \mathcal{A}, \mathcal{P})$? A partial answer to the last question is given in the following Th. 4.5. To this end remember the following.

Definition 4.4. A covering of a non empty set X is a subset $\mathcal{U}$ of $\mathcal{P}\left(X\right)$ so that $X = \cup\mathcal{U}$. It is said that the covering $\mathcal{U}$ of X is locally finite if any element of X belongs to a finite number of elements of $\mathcal{U}$. Further, a locally finite covering $\mathcal{U}$ of X is called bounded if

$$\eta = \sup\left\{\mathrm{card}\left\{U \in \mathcal{U} : x \in U\right\} : x \in X\right\} < \infty.$$

Then $\eta \in \mathbb{N}$ and we will say that η is the least upper bound of $\mathcal{U}$.

Theorem 4.5. *(i) Let $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a locally finite bounded covering of $\mathbb{N}$ with a least upper bound η. If $A \in \mathcal{B}(\mathrm{c}(\mathbb{N},\mathbb{C}))$ is defined by any infinite matrix $\{a_{n,m}\}_{n,m=0}^{\infty}$ so that $a_{n,m} = 0$ if $m \notin U_n$ then A is* ac*-regular for any random process on the Banach space sequence* $\mathrm{c}(\mathbb{N},\mathbb{C})$.

(ii) Let $A \in \mathcal{B}(\mathrm{c}(\mathbb{N},\mathbb{C}))$ induced by an infinite matrix of non negative coefficients $\{a_{n,m}\}_{n,m=0}^{\infty}$ with $a_{0,0} = 0$. Then A is L^r*- regular if* $1 \leqslant r < +\infty$.

Proof. (i) If $\{X_n\}_{n=1}^{\infty} \cup \{X\} \subseteq \mathcal{M}_{\mathrm{P}}(\Omega, \mathcal{A}, \mathbb{E})$ and $X = \text{ac-}\lim_{n\to\infty} X_n$ we know that $X = \text{ae-}\lim_{n\to\infty} X_n$ and by Corollary 2.3 it is defined a random process on $\mathrm{c}_0(\mathbb{N},\mathbb{C})$. If $n \in \mathbb{N}$ let $Y_n \triangleq \sum_{m=1}^{\infty} a_{n,m}(X_m - X)$. So, if $\varepsilon > 0$ then $\{|Y_n| \geqslant \varepsilon\} = \varnothing$ or

$$
\begin{aligned}
\{|Y_n| \geqslant \varepsilon\} &\subseteq \left\{ \sum_{m\in U_n} |a_{n,m}(X_m - X)| \geqslant \varepsilon \right\} \\
&\subseteq \left\{ \sup_{m\in U_n} |X_m - X| \sum_{m\in U_n} |a_{n,m}| \geqslant \varepsilon \right\} \\
&\subseteq \left\{ \sup_{m\in U_n} |X_m - X| \geqslant \varepsilon / \|A\| \right\} \\
&\subseteq \bigcup_{m\in U_n} \{|X_m - X| \geqslant \varepsilon / \|A\|\}.
\end{aligned}
$$

Consequently, if $N \in \mathbb{N}$ we estimate

$$
\begin{aligned}
\sum_{n=1}^{N} \mathrm{P}(\{|Y_n| \geqslant \varepsilon\}) &\leqslant \sum_{n=1}^{N} \sum_{m\in U_n} \mathrm{P}(\{|X_m - X| \geqslant \varepsilon / \|A\|\}) \\
&\leqslant \sum_{m\in \cup_{n=1}^{N} U_n} \mathrm{P}(\{|X_m - X| \geqslant \varepsilon / \|A\|\}) \operatorname{card}\{n : m \in U_n\} \\
&\leqslant \eta \sum_{m=1}^{\infty} \mathrm{P}(\{|X_m - X| \geqslant \varepsilon / \|A\|\}).
\end{aligned}
$$

Therefore,

$$
\sum_{n=1}^{\infty} \mathrm{P}(\{|Y_n| \geqslant \varepsilon\}) \leqslant \eta \sum_{m=1}^{\infty} \mathrm{P}(\{|X_m - X| \geqslant \varepsilon / \|A\|\}) < \infty
$$

and our claim follows.

(ii) Let $A \in \mathcal{B}(\mathrm{c}(\mathbb{N},\mathbb{C}))$ defined by an infinite matrix $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ with non negative coefficients and $a_{0,0} = 0$. Let $\{Z_m\}_{m=1}^{\infty} \cup \{Z\}$ be a sequence of random variables defining a Banach random process on $\mathrm{c}(\mathbb{N},\mathbb{C})$ so that $Z_m \xrightarrow{\mathrm{L}^r} Z$. Giving $n \in \mathbb{N}$ set $W_n \triangleq A_n(\{Z_m - Z\}_{m=1}^{\infty})$. Of course we may assume that $A \neq 0$. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_n)$ so that $\mu_n(S) \triangleq \|A\|^{-1} \sum_{m\in S} a_{n,m}$. Let us consider the function

$$F\colon \mathbb{N}\times\Omega\to \mathbb{C},\ F(m,w) \triangleq Z_m(w) - Z(w)\,.$$

Giving $\zeta \in \mathbb{C}$ and $r > 0$ it is easy to see that

$$\{|F-\zeta| < r\} = \bigcup_{m=1}^{\infty} \{m\} \times \{|Z_m - Z| < r\}\,,$$

i.e. $\{|F-\zeta| < r\}$ is clearly a measurable subset of $\mathbb{N}\times\Omega$ and since ζ and r are arbitrary F is measurable. Indeed, for almost all $w \in \Omega$ and $m \in \mathbb{N}$ there is a positive constant $K(w)$ so that $|Z_\upsilon(w)| \leqslant K(w)$ if $\upsilon \in \mathbb{N}$ and we have

$$\begin{aligned}\int_{\{1,\ldots,m\}} |F(\upsilon,w)|\, d\mu_n(\upsilon) &= \|A\|^{-1} \sum_{\upsilon=1}^{m} a_{n,\upsilon}\, |Z_\upsilon(w) - Z(w)| \\ &\leqslant 2K(w)\, \|A\|^{-1} \sum_{\upsilon=1}^{m} a_{n,\upsilon} \\ &\leqslant 2K(w).\end{aligned} \tag{4.3}$$

By an easy application of the monotone convergence theorem in (4.3) we deduce that $F(\circ,w) \in \mathcal{L}^1(\mathbb{N},\mu_n)$. Further,

$$F(\circ,w) = \lim_{m\to\infty} \sum_{\upsilon=1}^{m} (Z_\upsilon(w) - Z(w))\, \chi_{\{\upsilon\}}(\circ)$$

and if $m \in \mathbb{N}$ we have that

$$\left| \sum_{\upsilon=1}^{m} (Z_\upsilon(w) - Z(w))\, \chi_{\{\upsilon\}}(\circ) \right| \leqslant |F(\circ,w)|$$

on $\mathbb{N}$. By Lebesgue's dominated convergence theorem for almost all $w \in \Omega$ we get

$$\begin{aligned} W_n(w) &= \sum_{m=1}^{\infty} a_{n,m}\, (Z_m(w) - Z(w)) \\ &= \|A\| \sum_{m=1}^{\infty} (Z_m(w) - Z(w))\, \mu_n(\{m\}) \\ &= \|A\| \sum_{m=1}^{\infty} (Z_m(w) - Z(w)) \int_{\mathbb{N}} \chi_{\{m\}}(\upsilon)\, d\mu_n(\upsilon) \\ &= \|A\| \int_{\mathbb{N}} F(\upsilon,w)\, d\mu_n(\upsilon). \end{aligned} \tag{4.4}$$

Using (4.4) and applying the Minkowski's integral inequality we now write

$$\mathrm{E}(|W_n|^r)^{1/r} = \left(\int_\Omega |W_n(w)|^r\, d\,\mathrm{P}(w) \right)^{1/r} \tag{4.5}$$

$$
\begin{aligned}
&= \|A\| \left(\int_\Omega \left| \int_{\mathbb{N}} F(m,w)\, d\mu_n(m) \right|^r d\,\mathrm{P}(w) \right)^{1/r} \\
&\leqslant \|A\| \int_{\mathbb{N}} \left(\int_\Omega |F(m,w)|^r \, d\,\mathrm{P}(w) \right)^{1/r} d\mu_n(m) \\
&= \|A\| \int_{\mathbb{N}} \left(\int_\Omega |Z_m(w) - Z(w)|^r \, d\,\mathrm{P}(w) \right)^{1/r} d\mu_n(m) \\
&= \|A\| \int_{\mathbb{N}} \mathrm{E}\left(|Z_m - Z|^r\right)^{1/r} d\mu_n(m) \\
&= \sum_{m=1}^{\infty} a_{n,m}\, \mathrm{E}\left(|Z_m - Z|^r\right)^{1/r}.
\end{aligned}
$$

Finally, the sequence $\{\mathrm{E}(|Z_m - Z|^r)\}_{m=1}^{\infty}$ is bounded and the claim follows letting $n \to \infty$ in (4.5), using (4.1) and that $a_{0,0} = 0$. $\square$

4.2. $\mathcal{M}$-regularity on $[\mathcal{M}_{dt}([0,1], \mathcal{L}[0,1], \mathrm{C}[0,1]), \mathrm{l}^p(\mathrm{C}[0,1])]$

Theorem 4.6. *Let $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ be a disjoint bounded covering of $\mathbb{N}$ with a least upper bound η. Given $m \in \mathbb{N}$ let $n(m)$ be the unique positive integer so that $m \in U_{n(m)}$. Let $1 < p, q < \infty$ so that $1/p + 1/q = 1$ and let $a \triangleq \{a_{n,m}\}_{n,m=1}^{\infty}$ be a set of complex numbers so that the series $\sigma(a) \triangleq \sum_{m=1}^{\infty} \left|a_{n(m),m}\right|^q$ is finite. Given $x \in \mathrm{l}^p(\mathrm{C}[0,1])$ set*

$$A^a(x) = \left\{ \sum_{m\in U_n} a_{n,m} \cdot x_m \right\}_{n=1}^{\infty}.$$

Then
(i) $A^a(x) \in \mathrm{l}^p(\mathrm{C}[0,1])$.
(ii) $A^a \in \mathcal{B}[\mathrm{l}^p(\mathrm{C}[0,1])]$.
(iii) The class $\mathcal{R} \triangleq \{A^a : \sigma(a) < \infty\}$ is simply almost completely regular on

$$[\mathcal{M}_{dx}([0,1], \mathcal{L}[0,1], \mathrm{C}[0,1]), \mathrm{l}^p(\mathrm{C}[0,1])].$$

(iv) The class $\mathcal{R} \triangleq \{A^a : \sigma(a) < \infty\}$ is regular in the mean on any random process $\{X_n\}_{n=1}^{\infty} \cup \{X\}$ so that $\sum_{n=1}^{\infty} \|\mathrm{E}(X_n - X)\|_{\infty}^p < \infty$.

Proof. (i) Since $\mathcal{U}$ is a bounded covering of $\mathbb{N}$ then $A^a(x) \hookrightarrow \mathrm{C}[0,1]$ if $x \in \mathrm{l}^p(\mathrm{C}[0,1])$. Indeed, if $a \in \mathcal{R}$ and $N \in \mathbb{N}$ we obtain

$$
\begin{aligned}
\left[\sum_{n=1}^{N} \|A_n^a(x)\|_{\infty}^p \right]^{1/p} &\leqslant \left[\sum_{n=1}^{N} \left(\sum_{m\in U_n} |a_{n,m}|\, \|x_m\|_{\infty} \right)^p \right]^{1/p} \qquad (4.6) \\
&\leqslant \sum_{m\in U_1\cup\cdots\cup U_N} \|x_m\|_{\infty} \left(\sum_{n\in\mathbb{N}: m\in U_n} |a_{n,m}|^p \right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m \in \cup_1 \cup \cdots \cup U_N} \|x_m\|_\infty \cdot \left|a_{n(m),m}\right| \\
&\leqslant \sigma(a)^{1/q} \cdot \|x\|_{\mathrm{l}^p(\mathrm{C}[0,1])}
\end{aligned}
$$

Letting $N \to \infty$ from (4.6) we see that $A^a(x) \in \mathrm{l}^p(\mathrm{C}[0,1])$ and

$$
\|A^a(x)\|_{\mathrm{l}^p(\mathrm{C}[0,1])} \leqslant \sigma(a)^{1/q} \cdot \|x\|_{\mathrm{l}^p(\mathrm{C}[0,1])} .
$$

(ii) It is now clear that A^a is linear and that $\|A^a\| \leqslant \sigma(a)^{1/q}$.
(iii) Let $\{X_m\}_{m=1}^\infty \cup \{X\}$ be a random process of $\mathcal{M}_{dx}([0,1], \mathcal{L}[0,1], \mathrm{C}[0,1])$ on the Banach sequence space $\mathrm{l}^p(\mathrm{C}[0,1])$ so that $X_m \xrightarrow{a.c.} X$. Given $a \in \mathcal{R}$ we will show that $A_n^a(\{X_m - X\}_{m=1}^\infty) \xrightarrow{a.c} 0$. For, evidently we can assume $\sigma(a) > 0$. If $\varepsilon > 0$ and $n \in \mathbb{N}$ we write

$$
\begin{aligned}
\left\{\|A_n^a(\{X_m - X\}_{m=1}^\infty)\|_\infty \geqslant \varepsilon\right\} &= \left\{\left\|\sum_{m \in U_n} a_{n,m} \cdot (X_m - X)\right\|_\infty \geqslant \varepsilon\right\} \\
&\subseteq \left\{\sigma(a)^{1/q} \sum_{m \in U_n} \|X_m - X\|_\infty \geqslant \varepsilon\right\} \\
&\subseteq \bigcup_{m \in U_n} \left\{\|X_m - X\|_\infty \geqslant \frac{\varepsilon}{\sigma(a)^{1/q} \cdot \operatorname{card}(U_n)}\right\} \\
&\subseteq \bigcup_{m \in U_n} \left\{\|X_m - X\|_\infty \geqslant \frac{\varepsilon}{\sigma(a)^{1/q} \cdot \eta(a)}\right\}.
\end{aligned}
$$

Consequently, if $N \in \mathbb{N}$ we see that

$$
\begin{aligned}
\sum_{n=1}^N \int_0^1 \chi_{\left\{\|A_n^a(\{X_m - X\}_{m=1}^\infty)\|_\infty \geqslant \varepsilon\right\}} dt &\leqslant \sum_{n=1}^N \sum_{m \in U_n} \int_0^1 \chi_{\left\{\|X_m - X\|_\infty \geqslant \frac{\varepsilon}{\sigma(a)^{1/q} \cdot \eta(a)}\right\}} dt \\
&\leqslant \sum_{m=1}^\infty \int_0^1 \chi_{\left\{\|X_m - X\|_\infty \geqslant \frac{\varepsilon}{\sigma(a)^{1/q} \cdot \eta(a)}\right\}} dt < \infty,
\end{aligned}
$$

and our claim follows.
(iv) Let $X_n \xrightarrow{\mathrm{E}} X$, $a \in \mathcal{R}$. If $n \in \mathbb{N}$ and

$$
Y_n \triangleq A_n^a(\{X_m - X\}_{m=1}^\infty) \triangleq \sum_{m \in U_n} a_{n,m} \cdot (X_m - X)
$$

it will suffice to show that

$$
\sum_{n=1}^\infty \|\mathrm{E}(Y_n)\|_\infty^p < \infty. \tag{4.7}
$$

Indeed, we can assume $X = 0$ a.e. Thus, if $\upsilon \in \mathbb{N}$ and

$$
\|\phi_1\|_{\mathrm{BV}[0,1]} = \cdots = \|\phi_\upsilon\|_{\mathrm{BV}[0,1]} = 1
$$

we have

$$
\begin{aligned}
\left|\sum_{n=1}^{v}\langle \mathrm{E}\left(Y_n\right),\phi_n\rangle\right| &= \left|\sum_{n=1}^{v}\int_0^1\left(\int_0^1 Y_n(t)(s)d\phi_n\left(s\right)\right)dt\right| \\
&\left|\sum_{n=1}^{v}\int_0^1\left(\int_0^1\sum_{m\in U_n}a_{n,m}X_m\left(t\right)d\phi_n\left(s\right)\right)dt\right| \\
&\left|\sum_{n=1}^{v}\sum_{m\in U_n}a_{n,m}\langle \mathrm{E}(X_m),\phi_n\rangle\right| \\
&\leqslant \sum_{n=1}^{v}\sum_{m\in U_n}\left|a_{n,m}\right|\left\|\mathrm{E}(X_m)\right\|_\infty \\
&\leqslant \sum_{n=1}^{v}\left(\sum_{m\in U_n}\left\|\mathrm{E}(X_m)\right\|_\infty^p\right)^{1/p}\left(\sum_{m\in U_n}\left|a_{n,m}\right|^q\right)^{1/q} \\
&\leqslant \left(\sum_{n=1}^{\infty}\left\|\mathrm{E}\left(X_n\right)\right\|_\infty^p\right)^{1/p}\sigma\left(a\right)^{1/q}.
\end{aligned}
$$

But $\mathrm{l}^p\left(\mathrm{C}\left[0,1\right]\right)^* \approx \mathrm{l}^q\left(\mathrm{BV}\left[0,1\right]\right)$, where $\approx$ denotes an isometric isomorphism of Banach spaces. Therefore,

$$
\begin{aligned}
\left(\sum_{n=1}^{v}\left\|\mathrm{E}\left(Y_n\right)\right\|_\infty^p\right)^{1/p} &= \sup_{\|\phi_1\|_{\mathrm{BV}[0,1]}=\cdots=\|\phi_v\|_{\mathrm{BV}[0,1]}=1}\left|\sum_{n=1}^{v}\langle \mathrm{E}\left(Y_n\right),\phi_n\rangle\right| \\
&\leqslant \left(\sum_{n=1}^{\infty}\left\|\mathrm{E}\left(X_n\right)\right\|_\infty^p\right)^{1/p}\sigma\left(a\right)^{1/q},
\end{aligned}
$$

and (4.7) follows since v is arbitrary. □

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