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A purely geometric proof of the uniqueness of a triangle with given lengths of one side and two angle bisectors

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Abstract

We give a proof of triangle congruence on one side and two angle bisectors based on purely Euclidean geometry methods.

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MSC: 51M04, 51M05, 51M25

1. Introduction

In [1, 2] the uniqueness of a triangle with given lengths of one side and two angle bisectors was proven with the help of calculus methods. In this note we give a purely geometric proof of this fact.

2. The uniqueness of a triangle with given lengths of one side and two adjacent angle bisectors

Lemma 2.1. *Suppose that triangles ABC and A'B'C' have an equal side AB=A'B' and equal angle bisectors AL=A'L'. Let $\angle CAB < \angle C'A'B'$. Then AC<A'C'.*

Proof. Let $LB = KB, L'B' = K'B'$ (Figure 1). Then $\angle AKB = \angle ALC$ and $\Delta ACL \sim \Delta ABK$, $AC/AB = AL/AK$. Similarly $A'C'/A'B' = A'L'/A'K' = AL/A'K'$. Let $BN \perp AK, B'N' \perp A'K'$. $\angle CAB < \angle C'A'B'$, then $\angle LAB < \angle L'A'B'$ and so $AN > A'N'.AK = 2AN - AL > A'K' = 2A'N' - AL$. Then $AC/AB < A'C'/A'B'$ and $AC < A'C'$. □

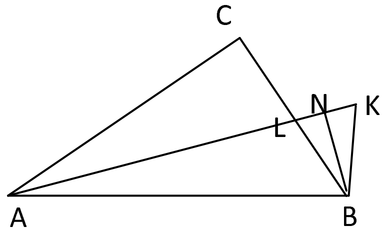

Figure 1

Theorem 2.2. *If one side and two adjacent angle bisectors of a triangle ABC are respectively equal to one side and two adjacent angle bisectors of a triangle A'B'C', then the triangles are congruent.*

Proof. Denote the two angle bisectors of ΔABC by AD and BE and let
$AD = A'D', BE = B'E', AB = A'B'$. If $\angle ABC = \angle A'B'C'$, then
$\angle ABE = \angle A'B'E' \Rightarrow \Delta ABE \cong \Delta A'B'E' \Rightarrow \angle BAC = \angle B'A'C' \Rightarrow$
$\Delta ABC \cong \Delta A'B'C'$.
Suppose that the triangles ABC and $A'B'C'$ have a common side AB and the adjacent angle bisectors of ΔABC are respectively equal to the adjacent angle bisectors of $\Delta A'B'C'$ $(AD = A'D', BE = B'E')$. We have to consider two cases.
Case 1. $\angle ABC > \angle A'B'C'$ and $\angle BAC > \angle B'A'C'$. Let us suppose that C' is in the interior of the triangle ACF (CF is the altitude of the triangle ACB) or C' is on CF, C' does not coincide with C (see Figure 2). We denote $K = AD \cap CF$ and

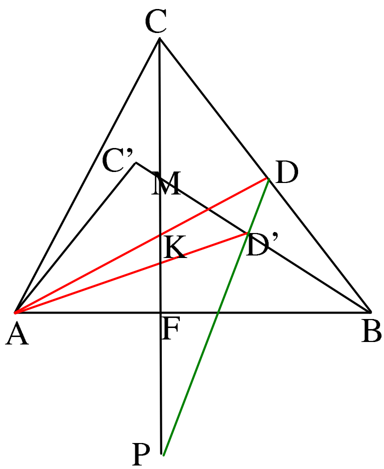

Figure 2

$M = C'B \cap CF$.

$$AC' < AC \Rightarrow \frac{AC}{AB} = \frac{CD}{DB} > \frac{AC'}{AB} = \frac{C'D'}{D'B} \geqslant \frac{MD'}{D'B} \quad ,$$

so $(DD') \cap (CF) = P$ and M is an interior point of interval CP. $\Delta DAD'$ is isosceles and therefore $\angle KD'P > 90°$, but $90° > \angle AKF > \angle KD'P$ and so we have a contradiction with $\angle KD'P > 90°$. So C' can not be in the interior of the triangle ACF or on CF. Similarly we get that C' can not be in the interior of the triangle BCF.
So the Case 1 is impossible.
Case 2. $\angle ABC < \angle A'B'C'$ and $\angle BAC > \angle B'A'C'$ (Figure 3).
We have $AC > AC'$ and $BC' > BC$ (Lemma 2.1). So $\angle CC'A > \angle ACC'$ and

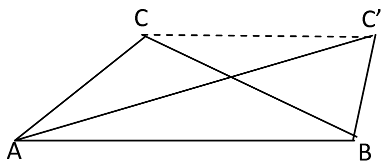

Figure 3

$\angle C'CB > \angle CC'B$. But $\angle ACC' > \angle C'CB$ and $\angle CC'B > \angle CC'A$. Then we again get a contradiction and this case is impossible too. □

3. The uniqueness of a triangle with given lengths of one side, one adjacent angle bisector and the opposite angle bisector

Lemma 3.1. *Suppose that triangles ABC and A'B'C' have an equal side AB=A'B' and equal angle bisector AL=A'L'. Let $\angle BAC < \angle B'A'C'$. Then BC<B'C'.*

Proof. By Lemma 2.1 we get $AC < A'C'$. Let $BH \perp AC$ and $B'H' \perp A'C'$ (Figure 4). So $AH > A'H'$and $BH < B'H'$. Then $CH = |AH - AC| < C'H' =$

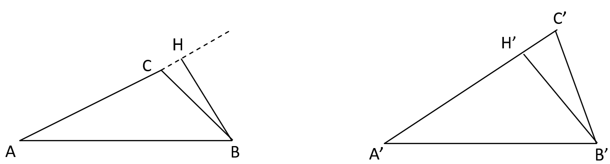

Figure 4

$|A'H' - A'C'|$ and so we have two right-angled triangles CHB and $C'H'B'$ with $CH < C'H'$ and $BH < B'H'$. Let $H'F = HC$ and $H'K = HB$ (Figure 5). So

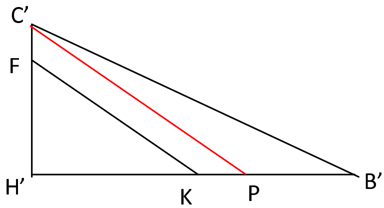

Figure 5

$FK = CB$. If $FK||C'B'$, then $FK < C'B'$. Suppose $\angle FKH' > \angle C'B'H'$. Let $C'P||FK$. Then $C'P > FK$. $\angle C'PB'$ is an obtuse angle and so $C'B' > C'P > FK = CB$. □

Theorem 3.2. *If one side, one adjacent angle bisector and the opposite angle bisector of a triangle ABC are respectively equal to one side, one adjacent angle bisector and the opposite angle bisector of a triangle A'B'C', then the triangles are congruent.*

Proof. Denote the two angle bisectors of triangles ABC and $A'B'C'$ by $AD, A'D'$ and $CE, C'E'$ correspondently and let $AD = A'D', CE = C'E', AB = A'B'$. Similarly to the proof of Theorem 2.2 we conclude that if $\angle BAC = \angle B'A'C'$ then the triangles are congruent. Let $\angle BAC < \angle B'A'C'$, then $A'C' > AC$ and $C'B' > CB$ (Lemma 2.1, 3.1). We prove that $C'E' > CE$. Let $\angle B"A'D' = \angle C"A'D' = \angle BAD, A'B" = AB, A'C" = AC$ (Figure 6), then $\Delta B"A'C' \cong \Delta BAC$ ($A'D'$ is a common angle bisector of the triangles $B'A'C'$ and $B"A'C"$).
We have to consider 3 cases.

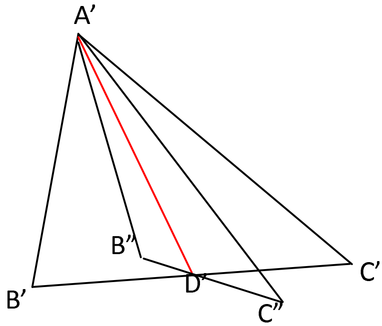

Figure 6

Case 1. Point $C"$ is in the interior of $\Delta C'A'D'$ (include interval $D'C'$).

In [3 ,Theorem 3] it was proven that in this case $C"E" = CE < C'E'$.
Case 2. Point $C"$ is in the exterior of $\Delta C'A'D'$ and $\angle A'C"B" = \angle ACB > \angle A'C'B'$. Let $C_1A' = CA$, $\angle A'C_1B_1 = \angle ACB$, $\angle C_1A'B_1 = \angle CA'B$ (Figure 7).
So $\Delta C_1A'B_1 \cong \Delta CAB$. According to [3, Lemma 1] the bisector of $\angle A'C_1B_1$

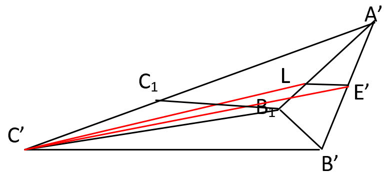

Figure 7

is less than the bisector of $\angle A'C'B_1$. Let $C'L$ be the triangle $A'C'B_1$ bisector. $\angle B_1A'C' < \angle B'A'C'$, so $C'B_1 < C'B'$ (the purely geometric proof of this fact was given in Euclid's Elements, Book 1, proposition 24). Then $B_1L/LA' = C'B_1/C'A' < C'B'/C'A' = B'E'/E'A'$. $A'B_1 = A'B'$ and so $\angle B_1LE'$ is an obtuse angle and $\angle C'LE' > \angle B_1LE' > 90°$. Then $C'E' > C'L > CE$.
Case 3. Point $C"$ is in the exterior of $\Delta C'A'D'$ and $\angle A'C"B" = \angle ACB < \angle A'C'B'$. Let $C'B_2 || C_1B_1$ and let $C'L_1$ be the angle bisector of the triangle $A'C'B_2$ (Figure 8). Then $C'L_1 > CE$. $C'B_2 < C'B'$ and again $B_2L_1/L_1A' = C'B_2/C'A' < C'B'/C'A' = B'E'/E'A'$, $\angle C'L_1E'$ is an obtuse angle and $C'E' > C'L_1 > CE$. □

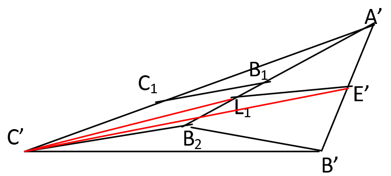

Figure 8

4. Notes

From each one of Theorem 2.1 and of [3,Theorem 3] the Steiner-Lehmus Theorem obviously follows and so these theorems provide its pure geometric proof.

References

[1] Oxman,V., On the existence of triangles with given lengths of one side and two adjacent angle bisectors, *Forum Geom.*, 4 (2004) 215–218.

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