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Limit theorems for the longest run

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Abstract

Limit theorems for the longest run in a coin tossing experiment are obtained.

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1. Introduction

Problems connected to the longest head run in a coin tossing experiment have been investigated for a long time. Erdős and Rényi (1970) proved for a fair coin that for arbitrary $0 < c_1 < 1 < c_2 < \infty$ and for almost all $\omega \in \Omega$ there exists a finite $N_0 = N_0(\omega, c_1, c_2)$ such that $[c_1 \operatorname{Log} N] \leqslant \mu(N) \leqslant [c_2 \operatorname{Log} N]$ if $N \geqslant N_0$ (here $\mu(N)$ denotes the length of the longest head run during the first N experiments, $[.]$ denotes the integer part, Log means logarithm of base 2). Erdős and Révész (1975) improved the above upper and lower bounds, moreover they proved other strong theorems for $\mu(N)$. Deheuvels (1985) Theorem 2 offers a.s. upper and lover bounds for the k-th longest head run for a biased coin. Földes (1979) studied the case of a fair coin and obtained limit theorems for the longest head run containing at most T tails. Binswanger and Embrechts (1994) gave a review of the results on the longest head run and their applications to gambling and finance. In the point of view of applications recursive algorithms for the distribution of the longest head run are important (see Kopociński (1991), Muselli (2000)). Fazekas and Noszály (2007) studied the limit distribution of the longest T-interrupted run of heads and recursive algorithms for the distribution in the case of a biased coin. Schilling (1990) gave an overview of limit theorems, algorithms and applications. Schilling (1990) studied pure head runs and runs of pure head or pure tails, too.

In this paper we study a coin tossing experiment. That is the underlying random variables are $X_1, X_2, \ldots$. We assume that $X_1, X_2, \ldots$ are independent and identically distributed with $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = q = 1 - p$. I.e. we write 1 for a head and 0 for a tail. In Section 2 we study pure runs, i.e. runs containing only heads or containing only tails. In Section 2 we prove limit theorems for the longest run. Our theorems 2.5–2.8 are versions of theorems 1–4 in Földes [9]. These are limit theorems for a fair coin. We consider the case of a biased coin in theorems 2.8 and 2.10.

Recently several papers are devoted to the study of almost sure limit theorems (see Berkes, Csáki (2001), Berkes, Dehling and Móri (1991), Fazekas and Rychlik (2002), Major (1998) and the references therein).

In Section 3 we obtain an almost sure limit theorem for the longest run (Theorem 3.1). We remark that for the longest run there is no limiting distribution (in Theorem 2.10 we give an accompanying sequence for it). However, for the logarithmic average we obtain limiting distribution. Our Theorem 3.1 is a version of Corollary 5.1 of Móri [16].

2. Limit theorems for longest runs

Consider N tossings of a coin. In this part we prove some limit theorems for longest runs. The theorems concern arbitrary pure runs (i.e. pure head runs or pure tail runs).

We shall use the next notation. Let $\xi(n, N) = \xi(n, N, \omega)$ denote the number of pure head sequences having length n.

Let $\xi^*(n, N) = \xi(n, N, \omega)$ denote the number of pure head or pure tail sequences having length n.

Let $\tilde{\xi}(n, N) = \tilde{\xi}(n, N, \omega)$ denote the number of disjoint pure head sequences with length being at least n.

Let $\tilde{\xi}^*(n, N) = \tilde{\xi}^*(n, N, \omega)$ denote the number of disjoint pure head or pure tail sequences with length being at least n.

Let $\tau(n) = \tau(n, \omega)$ denote the smallest number of casts which are necessary to get at least one pure head run of length n, that is

$$\tau(n) = \min\{N \mid \xi(n, N) > 0\}.$$

Let $\tau^*(n) = \tau^*(n, \omega)$ denote the smallest number of casts which are necessary to get at least one pure head or one pure tail run of length n, that is

$$\tau^*(n) = \min\{N \mid \xi^*(n, N) > 0\}.$$

Let $\mu(N) = \mu(N, \omega)$ denote the length of the longest pure head run in the first N trials, that is

$$\mu(N) = \max\{n \mid \xi(n, N) > 0\}.$$

Let $\mu^*(N) = \mu^*(N, \omega)$ denote the length of the longest pure head or pure tail run in the first N trials, that is

$$\mu^*(N) = \max\{n \mid \xi^*(n, N) > 0\}.$$

Here $\omega \in \Omega$, where $(\Omega, A, \mathbb{P})$ is the underlying probability space.

In this section we obtain analogues of Theorems $1-4$ in Földes [9] for arbitrary pure runs.

First consider the case of a fair coin. For convenience we quote the results of Földes.

Theorem 2.1 (Theorem 1 in [9]). *If $N \to \infty$ and $n \to \infty$ such that*

$$\frac{N}{2^{n+1}} \to \lambda > 0, \tag{2.1}$$

then we hawe

$$\lim_{N\to\infty} \mathbb{P}(\tilde{\xi}(n,N) = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \dots . \tag{2.2}$$

Theorem 2.2 (Theorem 2 in [9]). *Under the condition of Theorem 2.1 the distribution of $\xi(n,N)$ converges to a compound Poisson distribution, namely*

$$\mathbb{E}(z^{\xi(n,N)}) \to \exp\left(\lambda\left(\frac{(1-\frac{1}{2})z}{1-\frac{1}{2}z} - 1\right)\right). \tag{2.3}$$

Theorem 2.3 (Theorem 3 in [9]). *For $0 < x < \infty$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau(n)}{2^{n+1}} \leqslant x\right) = 1 - e^{-x}. \tag{2.4}$$

Theorem 2.4 (Theorem 4 in [9]). *For any integer k we have*

$$\mathbb{P}(\mu(N) - [\operatorname{Log} N] < k) = \exp(-2^{-(k+1-\{\operatorname{Log} N\})}) + o(1) \tag{2.5}$$

where $[a]$ denotes the integer part of a and $\{a\} = a - [a]$.

We use the next connection between the pure head runs and pure runs (see, for example, Schilling in [21]).

Remark 2.5. The next relation is true.

$$2\,\mathrm{card}\{\tilde{\xi}(n-1, N-1) = k\} = \mathrm{card}\{\tilde{\xi}^*(n,N) = k\}, \quad k = 0, 1, 2, \dots . \tag{2.6}$$

Theorem 2.6. *If $N \to \infty$ and $n \to \infty$ such that*

$$\frac{N}{2^{n+1}} \to \lambda > 0, \tag{2.7}$$

then we hawe

$$\lim_{N\to\infty} \mathbb{P}(\tilde{\xi}^*(n,N) = k) = \frac{e^{-2\lambda}(2\lambda)^k}{k!}, \quad k = 0, 1, 2, \dots . \tag{2.8}$$

Proof. If we use the (2.6.) connection we have for $k = 0, 1, 2, \ldots$

$$\mathbb{P}(\tilde{\xi}^*(n,N) = k) = \frac{\mathrm{card}\{\tilde{\xi}^*(n,N) = k\}}{2^N} = \frac{2\,\mathrm{card}\{\tilde{\xi}(n-1,N-1) = k\}}{2^N} = \\ = \mathbb{P}(\tilde{\xi}(n-1,N-1) = k).$$

If $\frac{N}{2^{n+1}} \to \lambda$, then

$$\frac{N-1}{2^{(n-1)+1}} = 2\frac{N-1}{N}\frac{N}{2^{n+1}} \to 2\lambda.$$

By Theorem 2.1, we obtain that

$$\lim_{n\to\infty} \mathbb{P}(\tilde{\xi}^*(n,N) = k) = e^{-2\lambda}\frac{(2\lambda)^k}{k!}.$$

This completes the proof of Theorem 2.5. □

Theorem 2.7. *Under the condition (2.1) the distribution of* $\xi^*(n,N)$ *converges to a compound Poisson distribution, namely*

$$\lim_{N\to\infty} \mathbb{E}(z^{\xi^*(n,N)}) = \exp\left(2\lambda\left(\frac{(1-\frac{1}{2})z}{1-\frac{1}{2}z} - 1\right)\right). \tag{2.9}$$

Proof. By (2.6.), we have

$$\mathbb{E}z^{\xi^*(n,N)} = \sum_{k=0}^{\infty} z^k \mathbb{P}(\xi^*(n,N) = k) = \sum_{k=0}^{\infty} z^k\,\mathrm{card}\{\xi^*(n,N) = k\}/2^N = \\ = \sum_{k=0}^{\infty} z^k 2\,\mathrm{card}\{\xi(n-1,N-1) = k\}/2^N = \sum_{k=0}^{\infty} z^k \mathbb{P}(\xi(n-1,N-1) = k) = \\ = \mathbb{E}z^{\xi(n-1,N-1)}.$$

By Theorem 2.2

$$\mathbb{E}z^{\xi(n-1,N-1)} = \exp\left(2\lambda\left(\frac{(1-\frac{1}{2})z}{1-\frac{1}{2}z} - 1\right)\right).$$

This completes the proof of Theorem 2.6. □

The next theorem state that the limit distribution of $\frac{\tau^*}{2^{n+1}}$ is exponential with parameter 2.

Theorem 2.8. *For* $0 < x < \infty$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^*(n)}{2^{n+1}} \leqslant x\right) = 1 - e^{-2x} \tag{2.10}$$

Proof. The theorem is the consequence of the calculation below and Theorem 2.3.

$$\begin{aligned}
&\mathbb{P}\left(\frac{\tau^*(n)}{2^n} > x\right) = \mathbb{P}(\text{ from } [2^n x] \text{ trials there is no run of lenght } n) = \\
&= \frac{\text{card}\{\text{ from } [2^n x] \text{ trials there is no run of lenght } n\}}{2^{[2^n x]}} = \\
&= \frac{2\ \text{card}\{\text{from } [2^n x]-1 \text{ trials there is no head run of lenght } n-1\ )}{2^{[2^n x]}} = \\
&= \mathbb{P}(\tau(n-1) > [2^n x]-1) = \mathbb{P}\left(\frac{\tau(n-1)}{2^n} > \frac{[2^n x]-1}{2^n}\right) = \\
&= \mathbb{P}\left(\frac{\tau(n-1)}{2^n} > x + a_n\right) = \mathbb{P}\left(\frac{\tau(n-1)}{2^n} - a_n > x\right)
\end{aligned}$$

where $\frac{[2^n x]-1}{2^n} = x + a_n$ and $a_n \to 0$. If we use Slutsky's theorem and Theorem 2.3, we get that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau(n-1)}{2^n} - a_n > x\right) = e^{-x}.$$

So

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^*(n)}{2^{n+1}} \leqslant x\right) = 1 - e^{-2x}.$$

This completes the proof of Theorem 2.7. □

Theorem 2.9. *For any integer k we have*

$$\mathbb{P}(\mu^*(N) - [\text{Log}(N-1)] < k) = \exp(-2^{-(k-\{\text{Log}(N-1)\})}) + o(1). \tag{2.11}$$

Proof. By Remark 2.1, we have

$$\begin{aligned}
&\mathbb{P}(\mu^*(N) - [\text{Log}(N-1)] < k) = \\
&= \frac{\text{card}\{\mu^*(N) - [\text{Log}(N-1)] < k\}}{2^N} = \\
&= 2\,\text{card}\{\mu(N-1) - [\text{Log}(N-1)] < k-1\}/2^N = \\
&= \mathbb{P}(\mu(N-1) - [\text{Log}(N-1)] < k) = \\
&= \exp\left(-2^{-(k-\{\text{Log}(N-1)\})}\right) + o(1),
\end{aligned}$$

where we applied Theorem 2.4. This completes the proof of Theorem 2.8. □

Now consider the case of a biased coin. Let p be the probability of tail and $q = 1 - p$ the probability of head. Let $V_N(p)$ denote the probability that the longest run in N trials is formed by heads. Then, by Theorem 5 of Musselli [19],

$$\lim_{N\to\infty} V_N(p) = \begin{cases} 0 & \text{if } 0 \leqslant p < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < p \leqslant 1. \end{cases} \tag{2.12}$$

Theorem 2.10. *Let $p > q$. For $0 < x < \infty$*

$$\lim_{n\to\infty} \mathbb{P}(\tau^*(n)qp^n \leqslant x) = 1 - e^{-x}. \tag{2.13}$$

Proof. We have

$$\lim_{n\to\infty} \mathbb{P}(\tau(n)qp^n \leqslant x) = 1 - e^{-x}. \tag{2.14}$$

(2.14) is mentioned in Móri [16] without proof and it is proved in Fazekas-Noszály [8]. Using (2.12), (2.14) implies (2.13). □

Theorem 2.11. *Let $p > q$. Let* Log *denote the logarithm of base $1/p$. Then for any integer k*

$$\mathbb{P}(\mu^*(N) - [\operatorname{Log} N] < k) = \exp(-qp^{k-\{\operatorname{Log} N\}}) + o(1). \tag{2.15}$$

Proof. By Gordon-Schilling-Waterman [11] or Fazekas-Noszály [8],

$$\mathbb{P}(\mu(N) - [\operatorname{Log} N] < k) = \exp(-qp^{k-\{\operatorname{Log} N\}}) + o(1). \tag{2.16}$$

(2.12) and (2.16) implies (2.15). □

3. An a.s. limit theorem for the longest run

In this part we prove an a.s. limit theorem for the longest run. Our theorem is a version of the following result of Móri. Let p be the probability of the head. Let Log denote the logarithm of base $1/p$. Let log denote the logarithm of base e.

Remark 3.1 (A particular case of Corollary 5.1 in Móri [16]).

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbb{I}(\mu(i) - \operatorname{Log} i < t) = \int_t^{t+1} \exp(-qp^z)dz \quad \text{a.s.} \tag{3.1}$$

Let us abbreviate $\mathbb{E}(\tau^*(n))$ by $E(n)$ and $\mathbb{P}(\tau^*(n) = n)$ by $p(n)$. To prove the a.s. limit theorem for the longest run we shall need the next results.

Remark 3.2 (See Lemma 2.2 in Móri [16]).

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^*(n)}{E(n)} > t\right) = e^{-t} \tag{3.2}$$

uniformly in $t \geqslant 0$.

Proposition 3.3 (A particular case of Theorem 3.1 in Móri [16]). *Suppose that f is a positive, increasing, differentiable function such that $E(m) \sim f(m)$ and the limit*

$$c = \lim_{t\to\infty} (\log f(t))' \tag{3.3}$$

exists. Let $g = f^{-1}$. Assume that $0 < c < \infty$. Then for every $t \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbb{I}(\mu^*(i)) - g(i) < t) = \int_0^1 F(c(t+z))dz \quad a.s., \tag{3.4}$$

where $F(z) = \exp(-\exp(-z))$.

The following result is the a.s. limit theorem for the longest run.

Theorem 3.4.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbb{I}(\mu^*(i) - \operatorname{Log} i < t) = \begin{cases} \int_t^{t+1} \exp\left[-\left(\frac{1}{2}\right)^y\right] dy & \text{if } p = \frac{1}{2} \\ \int_t^{t+1} \exp\left[-qp^y\right] dy & \text{if } p > \frac{1}{2} \end{cases}$$

almost sure.

Proof. We distinguish two cases. First let $p = 1/2$. By Theorem 2.7., $\frac{\tau^*(n)}{2^{n+1}}$ has exponential limit distribution with expectation $1/2$, that is $\mathbb{P}(\frac{\tau^*(n)}{2^n} > t) = e^{-t}$.

Now we verify that $\mathbb{E}\mu^*(n) \sim 2^n$. By Remark 3.2, $\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^*(n)}{E(n)} > t\right) = e^{-t}$. Using the convergence of types theorem (Theorem 2 in Section 10 of Gnedenko-Kolmogorov [10]), we obtain that $\frac{E(n)}{2^n} \to 1$, if $n \to \infty$.

So we can choose in Proposition 3.1 $f(x) = 2^x$, $g(x) = \operatorname{Log} x$. We obtain that $c = \lim_{t\to\infty}(\log f(t))' = \log 2 \in]0, \infty[$. Therefore we can apply Proposition 3.1.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbb{I}(\mu^*(i) - \operatorname{Log} i < t) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbb{I}(\mu^*(i) - g(i) < t) =$$

$$\int_0^1 \exp\left[-\exp(-c(t+z))\right] dz = \int_0^1 \exp\left[-\left(\frac{1}{2}\right)^{t+z}\right] dz = \int_t^{t+1} \exp\left[-\left(\frac{1}{2}\right)^{y}\right] dy.$$

Now let $p > 1/2$. By Theorem 2.9,

$$\lim_{n\to\infty} \mathbb{P}(\tau^*(n)qp^n > x) = e^{-x}. \tag{3.5}$$

By Remark 3.2,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^*(n)}{E(n)} > x\right) = e^{-x}. \tag{3.6}$$

So $\frac{E(n)}{(qp^n)^{-1}} \to 1$, if $n \to \infty$. Therefore $E(n) \sim (qp^n)^{-1}$. So $f(x) = q^{-1}p^{-x} = \frac{1}{q}\left(\frac{1}{p}\right)^x$. So $g(x) = \operatorname{Log} x + \operatorname{Log} q$ and $c = \log \frac{1}{p}$. This completes the proof of Theorem 3.4. □

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