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A note on integral clock triangles

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Abstract

Given two triangles with integer sides (a, b, c) and (a, b, d), and with the corresponding angles C and D such that $C \neq D$ and $C + D \neq \pi$, we show how to find a, b, c, d from any rational values of $\cos C$ and $\cos D$. For $C + D = \pi$ we show that solutions only exist for certain rational values of $\cos C$.

Keywords: elliptic curves, triangles, rational points.

MSC: 11D09, 11D25, 11Y50.

1. Introduction

In 2000, Petulante and Kaja [2] showed how to generate integer triangles with a specified value of one rational cosine. In 2007, Tengely [4] extended this to consider "clock triangle pairs" where two sides of both triangles are common. In this short note, we consider one aspect of this problem.

Let C and D be two angles of triangles, respectively with sides (a, b, c) and (a, b, d), where we suppose a and b are integers. The cosine rule implies that, if c and d are also integers, then $\cos C$ and $\cos D$ must be rational. Tengely, however, considered situations which included common angles such as $\pi/6$ and $\pi/4$. This gives sides which are possibly quadratic surds and led to the use of quadratic fields in the analysis.

The problem we consider is:

Given rational values for $\cos C$ and $\cos D$, find integer values for a, b, c, d, if possible.

To simplify the analysis, set $g = \cos C$ and $h = \cos D$. Thus we need to find integer solutions to

$$a^2 - 2gab + b^2 = c^2 \qquad a^2 - 2hab + b^2 = d^2.$$

If we define $y = c/b$ and $x = a/b$ the first equation is of the form $y^2 = x^2 - 2gx + 1$, which has the obvious solution $x = 0, y = 1$. The tangent at this point $y = 1 + Mx$ meets the curve again where $x = 2(M+g)/(1-M^2)$. Defining $z = d/b$, the second quadratic is $z^2 = x^2 - 2hx + 1$. Substituting the x value gives

$$w^2 = M^4 + 4hM^3 + (4hg+2)M^2 + (8g-4h)M + (4g^2 - 4hg + 1)$$

if we define $w = z(1-M^2)$.

The quartic, in this form, is birationally equivalent to an elliptic curve. Using the standard transformations, as described in Mordell [1], we find the elliptic curve E_{gh} is given by

$$E_{gh} : v^2 = u^3 + 2(1 - g\,h)u^2 + (g^2-1)(h^2-1)u$$

with the transformation

$$M = \frac{g(h^2-1) - hu + v}{(u - h^2 + 1)}$$

and thus, to find rational M and hence x, we need rational points (u, v) on these curves.

The curve E_{gh} has clearly 3 points of order 2 where $v = 0$, namely $u = 0, u = (g-1)(h+1), u = (g+1)(h-1)$, and these points are distinct if $g \neq h$. Substituting into the formula for M, gives $x = 0$ or $x = \infty$. Thus to find non-trivial solutions we must consider other points.

The relation for M has zero denominator if $u = h^2 - 1$, which we find gives a rational value $v = \pm(g-h)(h^2-1)$. Define the point $P = ((h^2-1), (g-h)(h^2-1)$, and the three order 2 points $T_1 = (0,0)$, $T_2 = ((g+1)(h-1), 0)$ and $T_3 = ((g-1)(h+1), 0)$. Note that $-P = ((h^2-1), -(g-h)(h^2-1))$.

Then $P + T_1 = ((g^2-1), -(g-h)(g^2-1))$. For this to be distinct from P, we need $g^2 \neq h^2$, implying $C \neq D$ or $C \neq \pi - D$. We have $M = -g$ so $x = 0$, but using the negative of the v value gives $M = (g^2 - gh - 2)/(g+h)$ leading to $x = 4(g+h)/(4-(g-h)^2)$. Since g and h are both in $(-1, 1)$, the denominator is always strictly positive, so if $g + h \neq 0$, the value of x is non-zero, but possibly negative, giving a solution of equation (1.1), but not real-life triangles.

As a numerical example, let $g = \cos C = 1/2$ and $h = \cos D = 1/3$, giving $x = a/b = 120/143$, suggesting $a = 120, b = 143$. This easily gives $c = 133$ and $d = 153$.

Looking at $P + T_2$ and $P + T_3$, we find trivial solutions or the above formula for x or its inverse but no new original solutions.

We now look at $2P$, which we find has $u = (g+h)^2/4$ with $v = (g+h)(4-(g-h)^2)/8$, which gives the above formula for x. Using the negative value of v, however, leads to the following ratio

$$x = \frac{4(g+h)(g^2+2gh-3h^2+4)(3g^2-2gh-h^2-4)}{(4-(g-h)^2))((g-h)^2+4(g+h-1))((g-h)^2-4(g+h+1))}$$

which gives $x = -2441880/865007$ for $g = 1/2$ and $h = 1/3$.

A Heron triangle must have integer area, which then forces the sines of the angles to be rational. Thus we can assume that $\cos C = (1-t^2)/(1+t^2), \sin C = 2t/(1+t^2)$ and $\cos D = (1-r^2)/(1+r^2), \sin D = 2r/(1+r^2)$. For example, $t = 1/2, r = 2/3$ give $g = 3/5, h = 5/13$, which lead to $a = 260, b = 261, c = 233, d = 289$ with the two triangles having areas 27144 and 31320.

2. $C + D = \pi$

The assumption might be that, if $C + D = \pi$ or $h = -g$, then no solutions exist. This is not true - they only exist for certain g values.

Putting $h = -g$ into equation (1.3) gives

$$v^2 = u^3 + 2(g^2+1)u^2 + (g^2-1)^2 u$$

and, if we define $g = m/n$ with $0 < m < n$ coprime integers, and $X = n^2 u, Y = n^3 v$, we have

$$Y^2 = X^3 + 2(m^2+n^2)X^2 + (m^2-n^2)^2 X.$$

The roots of the right hand side show that the curve has 3 torsion points of order 2, $(0,0), (-(m-n)^2, 0), (-(m+n)^2, 0)$.

For this curve, a point (P, Q) has the X-coordinate of $2(P, Q)$ equal to

$$\frac{(P^2 - (m^2-n^2)^2)^2}{4Q^2}$$

and so, if (P, Q) is of order 4 we must have $P = \pm(m^2 - n^2)$, giving the 4 points of order 4, $(m^2-n^2, \pm 2m(m^2-n^2)), (n^2-m^2, \pm 2n(m^2-n^2))$.

Putting all of these through the various transformations we get $x = a/b$ equal to 0 or ∞.

This set of torsion points shows that the torsion subgroup is either isomorphic to $\mathbb{Z}2 \times \mathbb{Z}4$ or to $\mathbb{Z}2 \times \mathbb{Z}8$. For the latter we need points of order 8. Since $0 < m < n$, we must have

$$\frac{(P^2 - (m^2-n^2)^2)^2}{4Q^2} = n^2 - m^2 = r^2$$

and by the Nagell-Lutz theorem r will be an integer, see Silverman and Tate [3].

Thus (r, m, n) must form a primitive Pythagorean triple, and so $r = 2st, m = s^2 - t^2, n = s^2 + t^2$ for coprime integers s, t. Substituting into the above relation reduces to the quartic equation

$$(P^2 - 8st(s^2+st+t^2)P + 16s^4t^4)\,(P^2 + 8st(s^2-st+t^2)P + 16s^4t^4) = 0$$

and, investigating the discriminant of the factors, we find that, for integer roots, we must have $s^2 + t^2 = \square$. Defining $s = e^2 - f^2$ and $t = 2ef$, we find the four roots, which give the X-coordinates of the points of order 8 as

1. $X = 16f^3e(f-e)(e+f)^3$,

2. $X = 16e^3f(e-f)(e+f)^3$,
3. $X = 16f^3e(e+f)(e-f)^3$,
4. $X = 16e^3f(e+f)(f-e)^3$.

Using these X values and both the corresponding positive and negative Y values, we find that they all lead to $a/b = \pm 1$. Thus we have a solution with a and b the same length. Now we have

$$g = \frac{e^4 - 6e^2f^2 + f^4}{(e^2+f^2)^2}$$

and we find that $a = b = e^2 + f^2, c = 4ef, d = 2(e^2 - f^2)$ is a solution as long as the values are positive.

For g not of this form, to find possible solutions, we must have further rational points on the curves, which means that the rank of the curve must be greater than 0. Note that the values $g^2 - 1$ and $h^2 - 1$ giving rational points from section 1 lead to the 4 points of order 4.

Running numerical experiments gives Table 1 for small (m, n) pairs. Thus, solutions clearly do not exist for all g. For $g = 5/6$, the generator of the curve leads to the lengths $72, 35, 47, 103$.

m	n	Rank	Generator
1	2	0	
1	4	0	
3	4	1	(49,-490)
1	5	0	
4	5	0	
1	6	1	(5,90)
5	6	1	(121/4,3025/8)

Table 1: Rank of curve for small m, n.

Further experimentation, using the Birch and Swinnerton-Dyer conjecture, see Wiles [5], to estimate the heights of curve generators, shows that for $0 < m < n < 100$ the largest height is predicted to occur for $g = 30/97$, with the height being 23.8 or 47.6, depending on the height normalization used.

Using some home-grown software, the generator is found to have

$$X = \frac{701477928878^2}{4786945163^2} \qquad Y = \frac{701477928878 \times 715985634093390721663175}{4786945163^3}$$

which lead to values of a, b, c, d all having roughly 40 digits. It should be noted that John Cremona's mwrank package finds this point in seconds.

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