

# Inclusion properties of the intersection convolution of relations

Judita Dască, Árpád Száz

Institute of Mathematics, University of Debrecen

*Submitted 9 November 2008; Accepted 25 February 2009*

## Abstract

For various relations  $F$  and  $G$  on one groupoid  $X$  with zero to another  $Y$ , we establish several simple, but important inclusions among the relations  $F$ ,  $G$ ,  $F * G$ ,  $F + G(0)$ , and  $F(0) + G$ .

The latter relations are given here by  $(F + G(0))(x) = F(x) + G(0)$ ,  $(F(0) + G)(x) = F(0) + G(x)$ , and

$$(F * G)(x) = \bigcap \{F(u) + G(v) : x = u + v, F(u) \neq \emptyset, G(v) \neq \emptyset\}$$

for all  $x \in X$ . The intersection convolution  $*$  allows of a natural generalization of the Hahn-Banach type extension theorems.

*Keywords:* Groupoids, binary relations, intersection convolution.

*MSC:* Primary 20L13; Secondary 46A22.

## 1. A few basic facts on relations and groupoids

A subset  $F$  of a product set  $X \times Y$  is called a relation on  $X$  to  $Y$ . For each  $x \in X$ , the set  $F(x) = \{y \in Y : (x, y) \in F\}$  is called the image of  $x$  under  $F$ , or the value of  $F$  at  $x$ .

Now, the set  $D_F = \{x \in X : F(x) \neq \emptyset\}$  may be naturally called the domain of  $F$ . Moreover, if in particular  $D_F = X$ , then we may say that  $F$  is a relation of  $X$  to  $Y$ , or that  $F$  is a total relation on  $X$  to  $Y$ .

In particular, a relation  $f$  on  $X$  to  $Y$  is called a function if for each  $x \in D_f$  there exists  $y \in Y$  such that  $f(x) = \{y\}$ . In this case, by identifying singletons with their elements, we may simply write  $f(x) = y$ .

If  $X$  is a set and  $+$  is a function of  $X^2$  to  $X$ , then the function  $+$  is called an operation in  $X$  and the ordered pair  $X(+) = (X, +)$  is called a groupoid even if  $X$  is void.

In this case, we may simply write  $x + y$  in place of  $+(x, y)$  for any  $x, y \in X$ . Moreover, we may also simply write  $X$  in place of  $X(+)$  whenever the operation  $+$  is clearly understood.

In practical applications, instead of groupoids, it is usually sufficient to consider only semigroups. However, several definitions and theorems on semigroups can be naturally extended to groupoids.

For instance, if  $X$  is a groupoid, then for any  $A, B \subset X$ , we may naturally write  $A + B = \{a + b : a \in A, b \in B\}$ . Moreover, we may also write  $x + A = \{x\} + A$  and  $A + x = A + \{x\}$  for any  $x \in X$ .

Note that if in particular  $X$  is a group, then we may also naturally write  $-A = \{-a : a \in A\}$  and  $A - B = A + (-B)$  for any  $A, B \subset X$ . Though, the family  $\mathcal{P}(X)$  of all subsets of  $X$  is only a semigroup with zero.

Now, if  $F$  and  $G$  are relations on a set  $X$  to a groupoid  $Y$ , then the pointwise sum  $F + G$  of  $F$  and  $G$  can be naturally defined such that  $(F + G)(x) = F(x) + G(x)$  for all  $x \in X$ .

Note that if in particular  $X$  is also a groupoid, then the above pointwise sum of the relations  $F$  and  $G$  may be easily confused with the global sum  $F \oplus G = \{(x + z, y + w) : (x, y) \in F, (z, w) \in G\}$ .

## 2. The most important additivity properties of relations

Analogously to the usual definition of superadditive functions, we may naturally consider the following

**Definition 2.1.** A relation  $F$  on one groupoid  $X$  to another  $Y$  is called superadditive if for any  $x, y \in X$  we have

$$F(x) + F(y) \subset F(x + y).$$

**Remark 2.2.** Note that thus  $F$  is superadditive if and only if  $F \oplus F \subset F$ . That is,  $F$  is a subgroupoid of  $X \times Y$ .

Moreover, if in particular  $F$  is a reflexive, superadditive relation of  $X$  to itself, then  $F$  is already a translation relation in the sense that  $x + F(y) \subset F(x + y)$  for all  $x, y \in X$ .

In addition to Definition 2.1, we may also naturally introduce the following

**Definition 2.3.** A relation  $F$  on one groupoid  $X$  to another  $Y$  is called

- (1) subadditive if  $F(x + y) \subset F(x) + F(y)$  for all  $x, y \in X$ ;
- (2) semi-subadditive if  $F(x + y) \subset F(x) + F(y)$  for all  $x, y \in D_F$ ;
- (3) quasi-subadditive if  $F(x + y) \subset F(x) + F(y)$  for all  $x, y \in X$  with either  $x \in D_F$  or  $y \in D_F$ .

**Remark 2.4.** Now, the relation  $F$  may, for instance, be naturally called quasi-additive if it is both superadditive and quasi-subadditive.

In [9], by calling a relation  $F$  on one group  $X$  to another  $Y$  quasi-odd if  $-F(x) \cap F(-x) \neq \emptyset$  for all  $x \in D_F$ , the second author has shown that a nonvoid, quasi-odd, superadditive relation is already quasi-additive.

As some obvious generalizations of the above definitions, we may also naturally introduce the following definitions.

**Definition 2.5.** A relation  $F$  on a groupoid  $X$  with zero to an arbitrary groupoid  $Y$  is called

(1) zero-superadditive if  $F(x) + F(0) \subset F(x)$  and  $F(0) + F(x) \subset F(x)$  for all  $x \in X$ ;

(2) zero-subadditive if  $F(x) \subset F(x) + F(0)$  and  $F(x) \subset F(0) + F(x)$  for all  $x \in X$ .

**Definition 2.6.** A relation  $F$  on a group  $X$  to a groupoid  $Y$  is called

(1) inversion-superadditive if  $F(x) + F(-x) \subset F(0)$  for all  $x \in X$ ;

(2) inversion-subadditive if  $F(0) \subset F(x) + F(-x)$  for all  $x \in X$ ;

(3) inversion-quasi-subadditive if  $F(0) \subset F(x) + F(-x)$  for all  $x \in D_F$ .

**Remark 2.7.** Note that, in the latter case, we also have  $F(0) \subset F(-x) + F(x)$  for all  $x \in D_F$ .

Namely, if  $F(0) \neq \emptyset$ , then by the inversion-quasi-subadditivity of  $F$  we also have  $F(-x) \neq \emptyset$ , and thus  $-x \in D_F$  for all  $x \in D_F$ .

### 3. The intersection convolution of relations

**Definition 3.1.** If  $X$  is a groupoid, then for any  $x \in X$  and  $A, B \subset X$ , we define

$$\Gamma(x, A, B) = \{(u, v) \in A \times B : x = u + v\}.$$

**Remark 3.2.** Now, in particular, we may simply write  $\Gamma(x) = \Gamma(x, X, X)$ . Thus,  $\Gamma$  is just the inverse relation of the operation  $+$  in  $X$ . Moreover, we have

$$\Gamma(x, A, B) = \Gamma(x) \cap (A \times B).$$

**Definition 3.3.** If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$ , then we define a relation  $F * G$  on  $X$  to  $Y$  such that

$$(F * G)(x) = \bigcap \{F(u) + G(v) : (u, v) \in \Gamma(x, D_F, D_G)\}$$

for all  $x \in X$ . The relation  $F * G$  is called the intersection convolution of the relations  $F$  and  $G$ .

**Remark 3.4.** If in particular  $F$  and  $G$  are relations of  $X$  to  $Y$ , then we may simply write

$$(F * G)(x) = \bigcap_{x=u+v} (F(u) + G(v)) = \bigcap \{F(u) + G(v) : (u, v) \in \Gamma(x)\}.$$

A particular case of Definition 3.3 was already considered in [6]. But, the following theorems have only been proved in [9].

**Theorem 3.5.** *If  $F, G, H,$  and  $K$  are relations on one groupoid  $X$  to another  $Y$  such that*

$$(1) D_H \subset D_F \text{ and } F(u) \subset H(u) \text{ for all } u \in D_H;$$

$$(2) D_K \subset D_G \text{ and } G(v) \subset K(v) \text{ for all } v \in D_K;$$

*then  $F * G \subset H * K$ .*

Now, as some immediate consequences of this theorem, we can also state

**Corollary 3.6.** *If  $F, G,$  and  $H$  are relations on one groupoid  $X$  to another  $Y$  such that  $D_H \subset D_F$  and  $F(u) \subset H(u)$  for all  $u \in D_H$ , then  $F * G \subset H * G$ .*

**Corollary 3.7.** *If  $F, G,$  and  $H$  are relations on one groupoid  $X$  to another  $Y$  such that  $D_H \subset D_G$  and  $G(v) \subset H(v)$  for all  $v \in D_H$ , then  $F * G \subset F * H$ .*

**Theorem 3.8.** *If  $F$  and  $G$  are relations on a group  $X$  to a groupoid  $Y$ , then for any  $x \in X$  we have*

$$\begin{aligned} (F * G)(x) &= \bigcap \{F(x - v) + G(v) : v \in (-D_F + x) \cap D_G\} = \\ &= \bigcap \{F(u) + G(-u + x) : u \in D_F \cap (x - D_G)\}. \end{aligned}$$

Hence, by using that  $-X + x = X$  and  $x - X = X$  for all  $x \in X$ , we can immediately get

**Corollary 3.9.** *If  $F$  and  $G$  are relations on a group  $X$  to a groupoid  $Y$ , then for any  $x \in X$  we have*

$$(1) (F * G)(x) = \bigcap_{v \in D_G} (F(x - v) + G(v)) \text{ whenever } F \text{ is total;}$$

$$(2) (F * G)(x) = \bigcap_{u \in D_F} (F(u) + G(-u + x)) \text{ whenever } G \text{ is total.}$$

Hence, it is clear that in particular we also have

**Corollary 3.10.** *If  $F$  and  $G$  are relations of a group  $X$  to a groupoid  $Y$ , then for any  $x \in X$  we have*

$$(F * G)(x) = \bigcap_{v \in X} (F(x - v) + G(v)) = \bigcap_{u \in X} (F(u) + G(-u + x)).$$

## 4. Convolutional inclusions for quite general relations

By using the corresponding definitions, we can easily prove the following

**Theorem 4.1.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $(F * G)(x) \subset (F + G(0))(x)$  for all  $x \in D_F$  if  $G(0) \neq \emptyset$ ;
- (2)  $(F * G)(x) \subset (F(0) + G)(x)$  for all  $x \in D_G$  if  $F(0) \neq \emptyset$ .

**Proof.** If  $x \in D_F$  and  $G(0) \neq \emptyset$ , then  $(x, 0) \in \Gamma(x, D_F, D_G)$ . Therefore,

$$\begin{aligned} (F * G)(x) &= \bigcap \{F(u) + G(v) : (u, v) \in \Gamma(x, D_F, D_G)\} \subset \\ &\subset F(x) + G(0) = (F + G(0))(x). \end{aligned} \quad \square$$

In addition to this theorem, it is also worth proving the following two theorems.

**Theorem 4.2.** *If  $F$  and  $G$  are relations of one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F \subset F + G(0)$  if  $0 \in G(0)$ ;
- (2)  $G \subset F(0) + G$  if  $0 \in F(0)$ .

**Proof.** If the condition of (1) holds, then

$$F(x) = F(x) + \{0\} \subset F(x) + G(0) = (F + G(0))(x)$$

for all  $x \in X$ . Therefore, the conclusion of (1) also holds.  $\square$

**Theorem 4.3.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F + G(0) \subset F$  if  $G(0) \subset \{0\}$ ;
- (2)  $F(0) + G \subset G$  if  $F(0) \subset \{0\}$ .

**Proof.** If the condition of (1) holds, then

$$(F + G(0))(x) = F(x) + G(0) \subset F(x) + \{0\} = F(x)$$

for all  $x \in X$ . Therefore, the conclusion of (1) also holds.  $\square$

Now, as an immediate consequence of the latter two theorems, we can also state

**Corollary 4.4.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F = F + G(0)$  if  $G(0) = \{0\}$ ;
- (2)  $G = F(0) + G$  if  $F(0) = \{0\}$ .

Moreover, as an immediate consequence of Theorems 4.1 and 4.3, we can also state

**Theorem 4.5.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $(F * G)(x) \subset F(x)$  for all  $x \in D_F$  if  $G(0) = \{0\}$ ;
- (2)  $(F * G)(x) \subset G(x)$  for all  $x \in D_G$  if  $F(0) = \{0\}$ .

Hence, it is clear that in particular we also have

**Corollary 4.6.** *If  $F$  is a relation on one groupoid  $X$  with zero to another  $Y$  such that  $F(0) = \{0\}$ , then  $(F * F)(x) \subset F(x)$  for all  $x \in D_F$ .*

## 5. Inclusions for zero-subadditive and zero-superadditive relations

In addition to Theorems 4.2 and 4.3, we can also easily prove the following two theorems.

**Theorem 5.1.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $F \subset F + G(0)$  if  $F$  is zero-subadditive and  $F(0) \subset G(0)$ ;
- (2)  $G \subset F(0) + G$  if  $G$  is zero-subadditive and  $G(0) \subset F(0)$ .

**Proof.** If the conditions of (1) hold, then

$$F(x) \subset F(x) + F(0) \subset F(x) + G(0) = (F + G(0))(x)$$

for all  $x \in X$ . Therefore, the conclusion of (1) also holds.  $\square$

**Theorem 5.2.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $F + G(0) \subset F$  if  $F$  is zero-superadditive and  $G(0) \subset F(0)$ ;
- (2)  $F(0) + G \subset G$  if  $G$  is zero-superadditive and  $F(0) \subset G(0)$ .

**Proof.** If the conditions of (1) hold, then

$$(F + G(0))(x) = F(x) + G(0) \subset F(x) + F(0) \subset F(x)$$

for all  $x \in X$ . Therefore, the conclusion of (1) also holds.  $\square$

Now, as an immediate consequence of the latter two theorems, we can also state

**Corollary 5.3.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $F = F + G(0)$  if  $F$  is zero-additive and  $F(0) = G(0)$ ;
- (2)  $G = F(0) + G$  if  $G$  is zero-additive and  $G(0) = F(0)$ .

Moreover, combining Theorems 4.3 and 4.2 with Theorems 5.1 and 5.2, respectively, we can also at once state the following two theorems.

**Theorem 5.4.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F = F + G(0)$  if  $F$  is zero-subadditive and  $F(0) \subset G(0) \subset \{0\}$ ;
- (2)  $G = F(0) + G$  if  $G$  is zero-subadditive and  $G(0) \subset F(0) \subset \{0\}$ .

**Theorem 5.5.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F = F + G(0)$  if  $F$  is zero-superadditive and  $0 \in G(0) \subset F(0)$ ;
- (2)  $G = F(0) + G$  if  $G$  is zero-superadditive and  $0 \in F(0) \subset G(0)$ .

On the other hand, as an immediate consequence of Theorems 4.1 and 5.2, we can also state

**Theorem 5.6.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $(F * G)(x) \subset F(x)$  for all  $x \in D_F$  if  $F$  is zero-superadditive and  $\emptyset \neq G(0) \subset F(0)$ ;
- (2)  $(F * G)(x) \subset G(x)$  for all  $x \in D_G$  if  $G$  is zero-superadditive and  $\emptyset \neq F(0) \subset G(0)$ .

Hence, it is clear that in particular we also have

**Corollary 5.7.** *If  $F$  is a zero-superadditive relation on a groupoid  $X$  with zero to an arbitrary one  $Y$  such that  $F(0) \neq \emptyset$ , then  $(F * F)(x) \subset F(x)$  for all  $x \in D_F$ .*

## 6. Convolutional inclusions for superadditive and semi-subadditive relations

In addition to Theorem 5.6, it is also worth proving the following.

**Theorem 6.1.** *If  $F$ ,  $G$ , and  $H$  are relations on one groupoid  $X$  to another  $Y$  and  $x \in D_F + D_G$  such that*

$$F(u) + G(v) \subset H(u + v)$$

for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ , then

$$(F * G)(x) \subset H(x).$$

**Proof.** By the above assumptions, it is clear that

$$\begin{aligned} (F * G)(x) &= \bigcap \{F(u) + G(v) : (u, v) \in \Gamma(x, D_F, D_G)\} \subset \\ &\subset \bigcap \{H(u + v) : (u, v) \in \Gamma(x, D_F, D_G)\} = \\ &= \bigcap \{H(x) : (u, v) \in \Gamma(x, D_F, D_G)\} = H(x). \quad \square \end{aligned}$$

Now, as an immediate consequence of this theorem, we can also state

**Corollary 6.2.** *If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$  and  $x \in D_F + D_G$ , then*

(1)  $(F * G)(x) \subset F(x)$  if  $F(u) + G(v) \subset F(u + v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ ;

(2)  $(F * G) \subset G(x)$  if  $F(u) + G(v) \subset G(u + v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ .

Hence, it is clear that in particular we also have

**Corollary 6.3.** *If  $F$  is a superadditive relation on one groupoid  $X$  to another  $Y$ , then  $(F * F)(x) \subset F(x)$  for all  $x \in D_F + D_F$ .*

Analogously to Theorem 6.1, we can also easily prove the following

**Theorem 6.4.** *If  $F$ ,  $G$ , and  $H$  are relations on one groupoid  $X$  to another  $Y$  and  $x \in D_G + D_H$  such that*

$$F(u + v) = G(u) + H(v)$$

for any  $u \in D_G$  and  $v \in D_H$  with  $x = u + v$ , then

$$F(x) = (G * H)(x).$$

Now, as an immediate consequence of this theorem, we can also state

**Corollary 6.5.** *If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$  and  $x \in D_F + D_G$ , then*

(1)  $F(x) = (F * G)(x)$  if  $F(u + v) = F(u) + G(v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ ;

(2)  $G(x) = (F * G)(x)$  if  $G(u + v) = F(u) + G(v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ .

Hence, it is clear that in particular we also have

**Corollary 6.6.** *If  $F$  is a semi-additive relation on one groupoid  $X$  to another  $Y$ , then  $F(x) = (F * F)(x)$  for all  $x \in D_F + D_F$ .*

Moreover, as a counterpart of Theorem 6.1, we can also prove the following

**Theorem 6.7.** *If  $F$ ,  $G$ , and  $H$  are relations on one groupoid  $X$  to another  $Y$ , then for any  $x \in X$  the following assertions are equivalent:*

(1)  $F(x) \subset (G * H)(x)$ ;

(2)  $F(u + v) \subset G(u) + H(v)$  for any  $u \in D_G$  and  $v \in D_H$  with  $x = u + v$ .



**Proof.** If (1) holds and  $u \in D_G$  and  $v \in D_H$  such that  $x = u + v$ , then

$$\begin{aligned} F(u + v) &= F(x) \subset (G * H)(x) = \\ &= \bigcap \{G(s) + H(t) : (s, t) \in \Gamma(x, D_G, D_H)\} \subset G(u) + H(v) \end{aligned}$$

since  $(u, v) \in \Gamma(x, D_G, D_H)$ . Thus, (2) also holds.

While, if (2) holds, then for any  $(u, v) \in \Gamma(x, D_G, D_H)$  we have

$$F(x) = F(u + v) \subset G(u) + H(v)$$

since  $u \in D_G$  and  $v \in D_H$  such that  $x = u + v$ . Hence, it is clear that

$$F(x) \subset \bigcap \{G(u) + H(v) : (u, v) \in \Gamma(x, D_G, D_H)\} = (G * H)(x).$$

Therefore, (1) also holds. □

Now, as an immediate consequence of this theorem, we can also state

**Corollary 6.8.** *If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$ , then for any  $x \in X$  we have:*

(1)  $F(x) \subset (F * G)(x)$  if and only if  $F(u + v) \subset F(u) + G(v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ ;

(2)  $G(x) \subset (F * G)(x)$  if and only if  $G(u + v) \subset F(u) + G(v)$  for any  $u \in D_F$  and  $v \in D_G$  with  $x = u + v$ .

Hence, it is clear that in particular we also have

**Corollary 6.9.** *If  $F$  is a relation on one groupoid  $X$  to another  $Y$ , then for any  $x \in X$  the following assertions are equivalent:*

(1)  $F(x) \subset (F * F)(x)$ ;

(2)  $F(u + v) \subset F(u) + F(v)$  for any  $u, v \in D_F$  with  $x = u + v$ .

## 7. Convolutional equalities for semi-subadditive and zero-superadditive relations

Now, as a useful characterization of semi-subadditivity, we can also state

**Theorem 7.1.** *If  $F$  is a relation on one groupoid  $X$  to another  $Y$ , then the following assertions are equivalent:*

(1)  $F \subset F * F$ ;

(2)  $F$  is semi-subadditive.

**Proof.** If (1) holds, then in particular for any  $u, v \in D_F$ , we have  $F(u + v) \subset (F * F)(u + v)$ . Hence, by using Corollary 6.9, we can infer that  $F(u + v) \subset F(u) + F(v)$ . Therefore, (2) also holds.

Conversely, if (2) holds and  $x \in X$ , then in particular for any  $u, v \in D_F$ , with  $x = u + v$ , we have  $F(u + v) \subset F(u) + F(v)$ . Hence, by using Corollary 6.9, we can infer that  $F(x) \subset (F * F)(x)$ . Therefore, (1) also holds. □

From this theorem, by using Corollaries 3.6 and 3.7, we can immediately derive

**Corollary 7.2.** *If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$ , then*

(1)  $F \subset F * G$  if  $F$  is semi-subadditive,  $D_G \subset D_F$ , and  $F(x) \subset G(x)$  for all  $x \in D_G$ ;

(2)  $G \subset F * G$  if  $G$  is semi-subadditive,  $D_F \subset D_G$ , and  $G(x) \subset F(x)$  for all  $x \in D_F$ .

Now, as an immediate consequence of Theorem 4.5 and Corollary 7.2, we can also state

**Theorem 7.3.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

(1)  $F = F * G$  if  $F$  is total and subadditive,  $F(x) \subset G(x)$  for all  $x \in D_G$ , and  $G(0) = \{0\}$ ;

(2)  $G = F * G$  if  $G$  is total and subadditive,  $G(x) \subset F(x)$  for all  $x \in D_F$ , and  $F(0) = \{0\}$ .

Hence, it is clear that in particular we also have

**Corollary 7.4.** *If  $F$  is a subadditive relation of one groupoid  $X$  with zero to another  $Y$  such that  $F(0) = \{0\}$ , then  $F = F * F$ .*

Moreover, as an immediate consequence of Theorem 5.6 and Corollary 7.2, we can also state

**Theorem 7.5.** *If  $F$  and  $G$  are relations of a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

(1)  $F = F * G$  if  $F$  is total, subadditive, and zero-superadditive,  $F(x) \subset G(x)$  for all  $x \in D_G$ , and  $\emptyset \neq G(0) \subset F(0)$ ;

(2)  $G = F * G$  if  $G$  is total, subadditive, and zero-superadditive,  $G(x) \subset F(x)$  for all  $x \in D_F$ , and  $\emptyset \neq F(0) \subset G(0)$ .

Hence, it is clear that in particular we also have

**Corollary 7.6.** *If  $F$  is a subadditive and zero-superadditive relation of a groupoid  $X$  with zero to an arbitrary one  $Y$ , then  $F = F * F$ .*

On the other hand, from Corollary 6.5, we can immediately get

**Theorem 7.7.** *If  $F$  and  $G$  are relations on one groupoid  $X$  to another  $Y$  such that  $X = D_F + D_G$ , then*

(1)  $F = F * G$  if  $F(u + v) = F(u) + G(v)$  for all  $u \in D_F$  and  $v \in D_G$ ;

(2)  $G = F * G$  if  $G(u + v) = F(u) + G(v)$  for all  $u \in D_F$  and  $v \in D_G$ .

Hence, it is clear that in particular we also have

**Corollary 7.8.** *If  $F$  is a semi-additive relation on one groupoid  $X$  to another  $Y$  such that  $X = D_F + D_F$ , then  $F = F * F$ .*

## 8. Convolutional inclusions for zero-subadditive and zero-superadditive relations

From Theorem 4.2, by using Corollaries 3.6 and 3.7, we can immediately get

**Theorem 8.1.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F * G \subset (F + G(0)) * G$  if  $0 \in G(0)$ ;
- (2)  $F * G \subset F * (F(0) + G)$  if  $0 \in F(0)$ .

Moreover, as an immediate consequence of Corollary 4.4, we can also state

**Theorem 8.2.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $G(0) = \{0\}$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $F(0) = \{0\}$ .

On the other hand, from Theorems 5.1 and 5.2, by using Corollaries 3.6 and 3.7, we can immediately get the following theorems.

**Theorem 8.3.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $F * G \subset (F + G(0)) * G$  if  $F$  is zero-subadditive and  $F(0) \subset G(0)$ ;
- (2)  $F * G \subset F * (F(0) + G)$  if  $G$  is zero-subadditive and  $G(0) \subset F(0)$ .

**Theorem 8.4.** *If  $F$  and  $G$  are relations of a groupoid  $X$  with zero to an arbitrary one  $Y$ , then*

- (1)  $(F + G(0)) * G \subset F * G$  if  $F$  is zero-superadditive and  $\emptyset \neq G(0) \subset F(0)$ ;
- (2)  $F * (F(0) + G) \subset F * G$  if  $G$  is zero-superadditive and  $\emptyset \neq F(0) \subset G(0)$ .

Now, as an immediate consequence of these theorems, we can also state

**Corollary 8.5.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to an arbitrary one  $Y$  such that  $F(0) = G(0) \neq \emptyset$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $F$  is zero-additive;
- (2)  $F * G = F * (F(0) + G)$  if  $G$  is zero-additive.

Moreover, as some immediate consequences of Theorems 5.4 and 5.5, we can also state

**Theorem 8.6.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $F$  is zero-subadditive and  $F(0) \subset G(0) \subset \{0\}$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $G$  is zero-subadditive and  $G(0) \subset F(0) \subset \{0\}$ .

**Theorem 8.7.** *If  $F$  and  $G$  are relations on one groupoid  $X$  with zero to another  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $F$  is zero-superadditive and  $0 \in G(0) \subset F(0)$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $G$  is zero-superadditive and  $0 \in F(0) \subset G(0)$ .

## 9. Convolutional inclusions for zero-additive and inversion-additive relations

In addition to Theorem 8.3, we can also prove the following

**Theorem 9.1.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$ , then*

- (1)  $F * G \subset (F + G(0)) * G$  if  $G$  is zero-subadditive;
- (2)  $F * G \subset F * (F(0) + G)$  if  $F$  is zero-subadditive.

**Proof.** If the condition of (1) holds, then

$$F(u) + G(v) \subset F(u) + G(0) + G(v) = (F + G(0))(u) + G(v)$$

for all  $u, v \in X$ . Therefore, for any  $x \in X$ , we have

$$\begin{aligned} (F * G)(x) &= \bigcap \{F(u) + G(v) : (u, v) \in \Gamma(x, D_F, D_G)\} \subset \\ &\subset \bigcap \{(F + G(0))(u) + G(v) : (u, v) \in \Gamma(x, D_F, D_G)\} = \\ &= \bigcap \{(F + G(0))(u) + G(v) : (u, v) \in \Gamma(x, D_{F+G(0)}, D_G)\} = \\ &= ((F + G(0)) * G)(x) \end{aligned}$$

provided that  $G(0) \neq \emptyset$ . Therefore, the conclusion of (1) also holds. Namely, if  $G(0) = \emptyset$ , then  $(F + G(0)) * G = \emptyset * G = X \times Y$ .  $\square$

Note that if  $G$  is zero-superadditive and  $G(0) \neq \emptyset$ , then just the converse inclusion holds. Therefore, we can also state the following

**Theorem 9.2.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$ , then*

- (1)  $(F + G(0)) * G \subset F * G$  if  $G$  is zero-superadditive and  $G(0) \neq \emptyset$ ;
- (2)  $F * (F(0) + G) \subset F * G$  if  $F$  is zero-superadditive and  $F(0) \neq \emptyset$ .

Now, as an immediate consequence of the above theorems, we can also state

**Corollary 9.3.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $G$  is zero-additive and  $G(0) \neq \emptyset$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $F$  is zero-additive and  $F(0) \neq \emptyset$ .

On the other hand, combining Theorems 8.1 with Theorems 9.2, we can also at once state the following

**Theorem 9.4.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$  with zero, then*

- (1)  $F * G = (F + G(0)) * G$  if  $G$  is zero-superadditive and  $0 \in G(0)$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $F$  is zero-superadditive and  $0 \in F(0)$ .

Moreover, combining Theorems 8.4 and 8.3 with Theorems 9.1 and 9.2, respectively, we can also at once state the following theorems.

**Theorem 9.5.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $F$  is zero-superadditive,  $G$  is zero-subadditive, and  $\emptyset \neq G(0) \subset F(0)$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $F$  is zero-subadditive,  $G$  is zero-superadditive, and  $\emptyset \neq F(0) \subset G(0)$ .

**Theorem 9.6.** *If  $F$  and  $G$  are relations on a groupoid  $X$  with zero to a semigroup  $Y$ , then*

- (1)  $F * G = (F + G(0)) * G$  if  $F$  is zero-subadditive,  $G$  is zero-superadditive, and  $F(0) \subset G(0) \neq \emptyset$ ;
- (2)  $F * G = F * (F(0) + G)$  if  $F$  is zero-superadditive,  $G$  is zero-subadditive, and  $G(0) \subset F(0) \neq \emptyset$ .

Finally, we note that in addition to Theorem 4.1, we can also prove the following

**Theorem 9.7.** *If  $F$  and  $G$  are relations on a group  $X$  to a semigroup  $Y$ , then*

- (1)  $F + G(0) \subset F * G$  if  $F$  is superadditive,  $G$  is inversion-quasi-subadditive and  $G \subset F$ ;
- (2)  $F(0) + G \subset F * G$  if  $G$  is superadditive,  $F$  is inversion-quasi-subadditive and  $F \subset G$ .

**Proof.** If  $x \in X$  and the conditions of (1) hold, then we can easily see that

$$\begin{aligned} (F + G(0))(x) &= F(x) + G(0) \subset F(x) + G(-v) + G(v) \subset \\ &\subset F(x) + F(-v) + G(v) \subset F(x - v) + G(v) \end{aligned}$$

for all  $v \in D_G$ . Hence, by using Theorem 3.8, we can infer that

$$(F + G(0))(x) \subset \bigcap \{F(x - v) + G(v) : v \in (-D_F + x) \cap D_G\} = (F * G)(x).$$

Therefore, the conclusion of (1) also holds. □

Now, as an immediate consequence of Theorems 4.1 and 9.7, we can also state

**Theorem 9.8.** *If  $F$  and  $G$  are relations on a group  $X$  to a semigroup  $Y$ , then*

(1)  $F * G = F + G(0)$  *if  $F$  is total and superadditive,  $G$  is inversion-quasi-subadditive and  $G \subset F$ ;*

(2)  $F * G = F(0) + G$  *if  $G$  is total and superadditive,  $F$  is inversion-quasi-subadditive and  $F \subset G$ .*

Hence, it is clear that in particular we also have

**Corollary 9.9.** *If  $F$  is a superadditive and inversion-quasi-subadditive relation of a group  $X$  to a semigroup  $Y$ , then*

$$F * F = F + F(0) \quad \text{and} \quad F * F = F(0) + F.$$

## References

- [1] BEG, I., Fuzzy multivalued functions, *Bull. Allahabad Math. Soc.*, 21 (2006), 41–104.
- [2] BOROS, Z. AND SZÁZ, Á., Reflexivity, transitivity, symmetry, and antisymmetry of the intersection convolution of relations, *Rostock. Math. Kolloq.*, 63 (2008), 55–62.
- [3] CZERWIK, S., Functional Equations and Inequalities in Several Variables, *World Scientific*, London, 2002.
- [4] GLAVOSITS, T. AND SZÁZ, Á., Pointwise and global sums and negatives of binary relations, *An. St., Univ. Ovidius Constanta*, 11 (2003), 87–94.
- [5] STRÖMBERG, T., The operation of infimal convolution, *Dissertationes Math.*, 352 (1996), 1–58.
- [6] SZÁZ, Á., The intersection convolution of relations and the Hahn–Banach type theorems, *Ann. Polon. Math.*, 69 (1998), 235–249.
- [7] SZÁZ, Á., Translation relations, the building blocks of compatible relators, *Math. Montisnigri*, 12 (2000), 135–156.
- [8] SZÁZ, Á., An extension of an additive selection theorem of Z. Gajda and R. Ger, *Tech. Rep., Inst. Math.*, Univ. Debrecen 8 (2006), 1–24.
- [9] SZÁZ, Á., The intersection convolution of relations on one groupoid to another, *Tech. Rep., Inst. Math.*, Univ. Debrecen 2 (2008), 1–22.
- [10] SZÁZ, Á., Relationships between the intersection convolution and other important operations on relations, *Math. Pannon.*, 20 (2009), 99–107.

**Judita Dascál**

**Árpád Száz**

Institute of Mathematics

University of Debrecen

H-4010 Debrecen, Pf. 12

Hungary

e-mails: [jdascal@math.klte.hu](mailto:jdascal@math.klte.hu)

[szaz@math.klte.hu](mailto:szaz@math.klte.hu)