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On the existence of triangle with given angle and opposite angle bisectors length*

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Abstract

Let us denote by l_a, l_c the lengths of angle bisectors on the sides BC and AB, respectively. We prove that for given positive $l_a,\ l_c$ and angle $\beta = ABC\angle$ there is a unique triangle.

Keywords: triangle, bisector

MSC: 51-99

It is known that for given lengths of three angle bisectors there is always a unique triangle [2], for an elementary proof, see [5]. In this note we consider the question of existence of a triangle with given angle and the lengts of two angle bisectors. The proposed question was motivated by the works of V. Oxman [3, 4], where the conditions for the existence of a triangle with given length of one side and two angle bisectors were studied.

Using methods of elementary calculus we prove the following theorem

Theorem. *Given positive $l_a,\ l_c,\ \beta < \pi$, there is a unique triangle ABC with $\beta = ABC\angle$ and lengths of bisectors of angles to the sides BC, AB equal to $l_a,\ l_c$.*

Proof. Recall that that in a triangle ABC with sidelengths a, b, c the bisector of angle $CAB\angle$ has length

$$l_a = \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)}. \tag{1}$$

We shall prove that for given $BC = 1$, $\beta = ABC\angle$ and $p = \frac{l_a}{l_c}$ there is a unique triangle. For simplicity, let us denote the side lengths AB and AC by x and y, respectively. See Figure 1.

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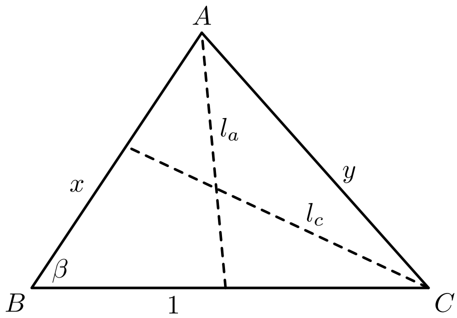

Figure 1

By the well-known Steiner-Lehmus theorem if a triangle has two equal bisectors ($p = 1$), then it is an isoscales triangle. Without lost of generality we may suppose $p > 1$.

From (1) we have

$$l_a^2 = xy\left(1 - \frac{1}{(x+y)^2}\right) \quad \text{and} \quad l_c^2 = y\left(1 - \frac{x^2}{(y+1)^2}\right).$$

Therefore

$$p^2 = \frac{l_a^2}{l_c^2} = \frac{x(x+y-1)(y+1)^2}{(x+y)^2(y+1-x)}.$$

Let us consider the function

$$f(x) = \frac{x(x+y-1)(y+1)^2}{(x+y)^2(y+1-x)}.$$

Note, y is a function of x, since by the law of cosines

$$y = \sqrt{x^2 + 1 - 2x\cos\beta}.$$

For convenience we ignore the dependence of y on x in notation. Our goal is to show that the equation $f(x) = p^2$ has a unique solution.

By the stronger form of Steiner-Lehmus theorem (see, e.g. [1])

$$l_a > l_c \iff a < c,$$

we immediately have that $x > 1$.
Obviously, $f(x)$ is a continuous function on the interval $[1, \infty)$. It is easy to check that

$$\lim_{x\to 1} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

By the above and the continuity of $f(x)$, Bolzano's theorem implies the existence of a solution of $f(x) = p^2$.

To prove the uniqueness we show that the function $f(x)$ is strictly increasing on $[1, \infty)$.
Since the derivative of the function y

$$y' = \frac{x - \cos\beta}{\sqrt{x^2 + 1 - 2x\cos\beta}}$$

is positive on $[1, \infty)$, hence y is strictly increasing throughout that interval.
Then

$$\frac{x + y - 1}{x + y} = 1 - \frac{1}{x + y} \tag{2}$$

is strictly increasing on $[1, \infty)$, too. Since

$$(y + 1 - x)' = y' - 1 = \frac{\cos^2\beta - 1}{(x - \cos\beta + \sqrt{x^2 + 1 - 2x\cos\beta})\sqrt{x^2 + 1 - 2x\cos\beta}}$$

is negative, we deduce that

$$\frac{1}{y + 1 - x} \tag{3}$$

strictly increases for $x \geqslant 1$.
Let

$$g(x) = \ln\frac{x(y + 1)^2}{x + y}.$$

Then

$$g'(x) = \frac{1}{x} + \frac{2y'}{y + 1} - \frac{1 + y'}{x + y}$$

which can be rewritten into the form

$$g'(x) = \left(y^2 + y + xyy' + xy'(2x - 1)\right)\frac{1}{x(y + 1)(x + y)}.$$

Clearly, $g'(x)$ is positive for any $x \geqslant 1$. From this follows that

$$\frac{x(y + 1)^2}{x + y} \tag{4}$$

is strictly increasing on $[1, \infty)$ (the positive function is strictly increasing if and only if its natural logarithm is strictly increasing). Taking into account that $f(x)$ is a product of functions (2–4) which strictly increase on the interval $[1, \infty)$, the assertion follows.

We have actually proved that for given $BC = 1$, $\beta = ABC\angle$ and $p = \frac{l_a}{l_c}$ there is a unique triangle. If two triangles are similar then their corresponding angle bisectors are proportionate. Using the similarity of triangles it can be easily deduced that for given l_a, l_c, β there is a unique triangle if and only if there exists a triangle for given $BC = 1$, β and $p = \frac{l_a}{l_c}$. □

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