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Crossed ladders and Euler's quartic

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Abstract

We investigate a particular form of the classical "crossed ladders" problem, finding many parametrized solutions, some polynomial, and some involving Fibonacci and Lucas sequences. We establish a connection between this particular form and a quartic equation studied by Euler, giving corresponding solutions to the latter.

MSC: 11D25, 11G05.

1. Introduction

The so-called "Crossed Ladders Problem" can be formulated as follows. Two ladders of lengths a,b, lean against two vertical walls as shown in Figure 1. The ladders cross each other at a point with distance c above the ground. Determine the distance x between the walls and the heights y, z, above the ground of the points where the ladders touch the walls.

The defining system of equations is

$$x^2 = a^2 - y^2 = b^2 - z^2, \tag{1.1}$$

$$c = \frac{yz}{y+z}. \tag{1.2}$$

There is enormous literature on the crossed ladder problem, as may be seen for example by consulting the extensive bibliography of Singmaster's "Sources in Recreational Mathematics", Section 6L: "Geometric Recreations. Crossed ladders"; see Singmaster [9]. The second and third authors have also recently investigated this problem; see Høibakk et al. [4, 5].

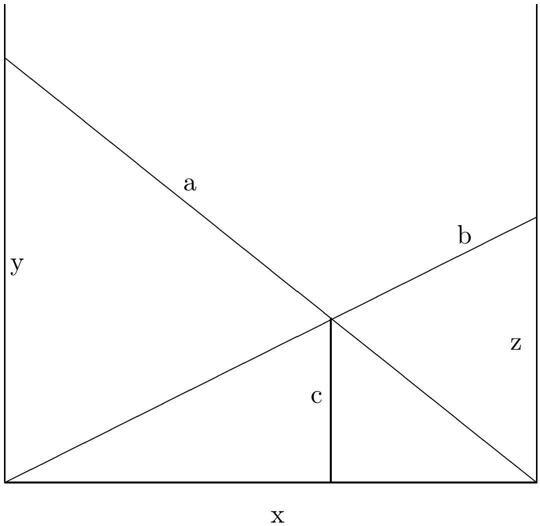

Figure 1: The crossed ladders

Our interest is in ladder problems where the lengths are all integers, and so we are reduced to finding integer solutions to the Diophantine system (1.1) where, without loss of generality, we may assume x, y, z, a, b have no common divisor. For certainly a solution of the ladder problem gives rise to such x, y, z, a, b; and conversely, given coprime x, y, z, a, b, then scaling by the factor

$$S = \frac{y+z}{\gcd(yz, y+z)}$$

results in an integer solution to (1.1), (1.2). Høibakk et al. [4] observe empirically that the ratio $y : z$ frequently takes an integer value greater than 1, and in this note we investigate the conditions and implications that imposing this restriction implies. Putting

$$y = Mz, \qquad M > 1, \tag{1.3}$$

gives

$$x^2 + M^2z^2 = a^2, \qquad x^2 + z^2 = b^2, \tag{1.4}$$

which as the intersection of two quadrics in projective 3-space (with a point at $z = 0$) represents an elliptic curve. A quartic form is easy to derive. At (1.3), (1.4) we can set

$$x : y : z : a : b = (X^2 - Y^2) : 2MXY : 2XY : Z : (X^2 + Y^2), \tag{1.5}$$

with

$$X^4 + (4M^2 - 2)X^2Y^2 + Y^4 = Z^2. \tag{1.6}$$

The inverse transformation is given by

$$\left(\frac{X}{Y}, \frac{Z}{Y^2}\right) = \left(\frac{x+b}{z}, \frac{2a(x+b)}{z^2}\right).$$

Thus solutions to the crossed ladder problem under the restriction (1.3) correspond to solutions of the Diophantine equation (1.6). The equation

$$X^4 + mX^2Y^2 + Y^4 = \square \tag{1.7}$$

has been studied since the 17th century. A solution is said to be trivial if either $XY = 0$ or if $X^2 = Y^2 = 1$, which can occur only when m is of the form $k^2 - 2$. Fermat showed there are no non-trivial solutions for $m = 0$. Euler showed that for $m = 14$ there are only trivial solutions, and found non-trivial solutions for 47 values of m between 2 and 200, and for 73 values of $-m$ between 2 and 200. Pocklington [7], Sinha [10], and Zhang [11] produced classes of m for which (1.7) has no non-trivial solutions; and Brown [3] completed the determination of solvability of (1.7) in the range $0 \leqslant m \leqslant 100$. Bremner & Jones [2] studied the equation in some detail, determining solvability of (1.7) (subject to standard conjectures) in the range $|m| \leqslant 3000$. We can deduce from the tables of [2], for example, that the smallest value of M for which non-trivial solutions of (1.6) exist is $M = 7$, with small solutions at $(X, Y) = (5, 1)$, leading to $(x, y, z, a, b, c) = (12, 35, 5, 74, 26, \frac{35}{8})$, and $(X, Y) = (6, 1)$ with $(x, y, z, a, b, c) = (35, 84, 12, 91, 37, \frac{21}{2})$.

It is our intention here to investigate the quartic (1.6) and derive parametrized families for M for which there exist non-trivial solutions. Surprisingly many values of M for which non-trivial solutions to (1.6) exist turn out to be members of such infinite families. We describe several such families, and indicate the corresponding point on (1.6). It is then straightforward to compute the corresponding solution to the crossed ladder problem by means of the ratios at (1.5). Some of the parametric families are given in terms of polynomials, others in terms of Fibonacci and Lucas sequences.

The ideas are essentially ad hoc, and by no means exhaustive: many values of M for which non-trivial solutions to (1.6) exist have not been found as members of infinite families.

We note that a cubic form of the elliptic curve at (1.4) is given by

$$E : v^2 = u^3 + (-2M^2 + 1)u^2 + (M^4 - M^2)u = u(u - M^2)(u - (M^2 - 1)), \tag{1.8}$$

where the maps are given by

$$x : z : a : b = (v^2 - u^2) : 2uv : (-u^3 + (M^4 - M^2)u) : (v^2 + u^2),$$

and

$$(u, v) = \left(\frac{(a-b)(a-x)}{z^2}, \frac{(a-b)(b+x)(a-x)}{z^3}\right).$$

We therefore also obtain the maps between the cubic curve at (1.8) and the quartic curve at (1.6), namely

$$(X, Y, Z) = (v,\ u,\ u^3 - (M^4 - M^2)u),$$

$$(u,v) = \left(\frac{X^2 + (2M^2-1)Y^2 + Z}{2Y^2},\ \frac{X(X^2 + (2M^2-1)Y^2 + Z)}{2Y^3} \right).$$

2. Linear parametrizations

We show that there can be no nontrivial points on the curve (1.6), where X, Y, Z are linear polynomials in M. The curve (1.8) represents a *rational* elliptic surface S, and as such we know by results of Shioda (see, for example, Shioda [8, Cor. 5.3, Thm. 10.8] that the Mordell-Weil group of (1.8) over $\mathbb{C}(M)$ is generated by those points which are given by polynomials u at most quadratic in M. Finding points on (1.8) over $\mathbb{C}(M)$ whose u-coordinate is at most quadratic in M is a straightforward machine computation. However, it is not necessary to carry out: the discriminant of the cubic model at (1.8) is equal to $16M^4(M^2-1)^2$, so that the curve is singular at $(M), (1/M), (M \pm 1)$. Computing the Kodaira reduction types, the Shioda formula for the rank gives $\mathrm{rank}(E(\mathbb{C}(M))) = 0$, and consequently $\mathrm{rank}(E(\mathbb{Q}(M))) = 0$. The only points on (1.8) are the torsion points, namely $(0,0)$, $(M^2, 0)$, and $(M^2-1, 0)$, corresponding to trivial points on (1.6).

3. Parametrizations of higher degree

At (1.4), we set without loss of generality $b+z = gp^2,\ b-z = gq^2,\ a+y = hr^2,\ a-y = hs^2,\ (x =)gpq = hrs,\ (p,q) = (r,s) = 1$, for integers p, q, r, s and g, h; and the restriction (1.3) demands

$$M = \frac{pq(r^2-s^2)}{rs(p^2-q^2)} \in \mathbb{Z}. \tag{3.1}$$

We correspondingly have crossed ladder solution

$$x : y : z : a : b = 2pq : M(p^2-q^2) : (p^2-q^2) : \frac{pq(r^2+s^2)}{rs} : (p^2+q^2),$$

and point at (1.6) given by

$$(X, Y, Z) = \left(p+q,\ p-q,\ \frac{2pq(r^2+s^2)}{rs} \right).$$

We study several particular cases.

3.1. Case I

We suppose $(r,s) = (pq, 1)$, which implies $g = h$ and demands

$$M = \frac{p^2q^2-1}{p^2-q^2} \in \mathbb{Z}. \tag{3.2}$$

An immediate family of solutions arises on setting $(p, q) = (2n+1, 2n-1)$ with corresponding M and point on (1.6) given by:

$$M = n(2n^2 - 1), \qquad (X, Y, Z) = (2n,\ 1,\ 8n^4 - 4n^2 + 1).$$

This gives numerical values of $M = \boxed{14, 51, 124, 245, \ldots}$

The curve E takes the form

$$E_n : v^2 = u\left(u - n^2(2n^2-1)^2\right)\left(u - (n^2-1)(4n^4+1)\right),$$

and the corresponding point on the elliptic curve E_n is

$$P(u, v) = \left((n^2-1)(2n^2-1)^2,\ 2n(n^2-1)(2n^2-1)^2\right).$$

We can compute multiples of P to obtain parametrized solutions to the crossed ladder problem of increasing degree. For example,

$$2P = \left(\frac{(8n^4 - 4n^2 + 1)^2}{16n^2},\ -\frac{(16n^4-1)(8n^4-4n^2+1)}{64n^3}\right),$$

corresponding to the crossed ladder solution

$$\begin{aligned}(x, y, z, a, b) = ((4n^2 - 2n - 1)(4n^2 + 2n - 1)(8n^3 - 2n - 1)(8n^3 - 2n + 1),\\ 8n^2(2n^2-1)(16n^4-1)(8n^4-4n^2+1),\\ 8n(16n^4-1)(8n^4-4n^2+1),\\ 2048n^{12} - 2048n^{10} + 896n^8 - 384n^6 + 128n^4 - 16n^2 + 1,\\ -(1024n^{10} - 768n^8 + 512n^6 - 160n^4 + 16n^2 + 1)).\end{aligned}$$

(Remark: the torsion group on E_n is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and points $Q + T$ for T torsion return the same values of (x, y, z, a, b) as for Q, up to sign; so it is only of interest to consider (x, y, z, a, b) corresponding to the direct multiples of P on E_n). The point $3P$ returns polynomials of degree 24.

We can obtain families of solutions to (3.2) by demanding that

$$n(pq + \epsilon) = p^2 - q^2, \qquad \epsilon = \pm 1, \tag{3.3}$$

for integers n. When $n = 1$, the theory of the Pell equation gives all solutions of (3.3) as $(p, q) = (F_{k+1}, F_k)$, where F_i is the i-th Fibonacci number. In this instance, $M = F_{k+1}F_k - (-1)^k$, with associated ladder solution

$$(x, y, z, a, b) = (2F_{k+1}F_k,\ F_{k+1}^2F_k^2 - 1,\ F_{k+1}^2 - F_k^2,\ F_{k+1}^2F_k^2 + 1,\ F_{k+1}^2 + F_k^2),$$

and point on (1.6)

$$M = F_{k+1}F_k - (-1)^k, \qquad (X, Y, Z) = (F_{k+2},\ F_{k-1},\ 2(F_{k+1}^2F_k^2 + 1)).$$

Numerical values of M that occur are $\boxed{M = 7, 14, 41, 103, 274, \ldots}$

When $n > 1$, solutions are provided in terms of the recurrence relation

$$R_i = nR_{i-1} + R_{i-2}, \quad i \geqslant 2, \qquad R_0 = \epsilon, R_1 = 1, \quad \epsilon = \pm 1.$$

Then

$$R_{k+1}^2 - R_k^2 = n(R_{k+1}R_k - \epsilon(-1)^k),$$

and taking $(p, q) = (R_{k+1}, R_k)$ gives rise to

$$M = \frac{1}{n}(R_{k+1}R_k + \epsilon(-1)^k).$$

We have

$$R_{2i+1} \equiv 1 \bmod n, \qquad R_{2i} \equiv \epsilon \bmod n,$$

so M will be integral precisely when k is odd. Thus setting $(p, q) = (R_{2i}, R_{2i-1})$ gives rise to

$$M = \frac{1}{n}(R_{2i}R_{2i-1} - \epsilon) \in \mathbb{Z}.$$

The corresponding point on (1.6) is given by

$$(X, Y, Z) = (R_{2i} + R_{2i-1},\ R_{2i} - R_{2i-1},\ 2(R_{2i}^2 R_{2i-1}^2 + 1)).$$

The case $n = 2$, $\epsilon = 1$, gives the well-known Pell sequence $R = \{1, 1, 3, 7, 17, 41, \ldots\}$ with corresponding M equal to $\boxed{59, 2029, \ldots}$. When $\epsilon = -1$, then the R-sequence is $\{-1, 1, 1, 3, 7, 17, 41, \ldots\}$ with corresponding M equal to $\boxed{11, 349, \ldots}$.

If we leave n as parameter, then we obtain the following values of M:

$$n^4 + 2\epsilon n^3 + 4n^2 + 4\epsilon n + 3 \tag{3.4}$$

$$n^8 + 2\epsilon n^7 + 8n^6 + 12\epsilon n^5 + 21n^4 + 22\epsilon n^3 + 20n^2 + 12\epsilon n + 5 \tag{3.5}$$

$$\vdots$$

Without loss of generality (by changing the sign of n if necessary) we may take $\epsilon = 1$, and the first line above corresponds to the crossed ladder problem solution given by

$$\begin{aligned}
&(x, y, z, a, b) = \\
&(2(n^2 + n + 1)(n^3 + n^2 + 2n + 1), \\
&n(n + 1)(n^2 + 1)(n^2 + n + 2)(n^4 + 2n^3 + 4n^2 + 4n + 3), \\
&n(n + 1)(n^2 + 1)(n^2 + n + 2), \\
&n^{10} + 4n^9 + 12n^8 + 24n^7 + 38n^6 + 46n^5 + 44n^4 + 32n^3 + 17n^2 + 6n + 2, \\
&n^6 + 2n^5 + 6n^4 + 8n^3 + 9n^2 + 6n + 2),
\end{aligned}$$

with point on (1.6) given by

$$(X, Y, Z) = ((n + 1)(n^2 + n + 2),\ n(n^2 + 1),$$

$$2(n^{10}+4n^9+12n^8+24n^7+38n^6+46n^5+44n^4+32n^3+17n^2+6n+2)).$$

Other approaches to making the quotient (3.2) integral include setting $p = F_{n+1} + F_{n-1}$, $q = F_n$, when we obtain

$$M = \frac{1}{4}(F_{n+2}+F_n)(F_n+F_{n-2}),$$

which is integral precisely when $n \equiv \pm 2 \bmod 6$. Numerical values of M are given by $M = \boxed{11, 551, \ldots}$. The point on (1.6) is

$$(X, Y, Z) = (F_{n+1}, F_{n-1}, \frac{1}{2}F_{2n+1}F_{2n-1}).$$

We can also take $(p, q) = (P_n, Q_n)$, where P_n, Q_n are the familiar Pell-sequences defined by $P_n = 2P_{n-1} + P_{n-2}$, $n \geqslant 2$, $P_0 = 1$, $P_1 = 1$, and $Q_n = 2Q_{n-1} + Q_{n-2}$, $n \geqslant 2$, $Q_0 = 0$, $Q_1 = 1$. Thus P is the sequence $1, 1, 3, 7, 17, 41, 99, \cdots$, Q the sequence $0, 1, 2, 5, 12, 29, 70, \cdots$, and $P_n^2 - 2Q_n^2 = (-1)^n$. Then

$$M = 2Q_n^2 - (-1)^n = P_{n+1}P_{n-1},$$

with numerical values $M = \boxed{7, 51, 287, \ldots}$. The point on (1.6) is

$$(X, Y, Z) = (P_n + Q_n, P_n - Q_n, 2(P_n^2 Q_n^2 + 1)) = (Q_{n+1}, Q_{n-1}, 2(P_n^2 Q_n^2 + 1)).$$

3.2. Case II

We assume $p^2 - q^2 = r^2 - s^2$, $q = rs$, which implies $M = p$, and demands

$$p^2 - r^2(s^2+1) = -s^2. \tag{3.6}$$

We consider s to be a fixed parameter, and by considering norms from the quadratic field $\mathbb{Q}(\sqrt{s^2+1})$ in which we note $s + \sqrt{s^2+1}$ is a unit of norm -1, can define solutions (p_i, r_i) by means of

$$p_i + r_i\sqrt{s^2+1} = (s + \sqrt{s^2+1})^{2i}(p_0 + r_0\sqrt{s^2+1})$$

for an initial solution (p_0, r_0). It is readily seen that p_i and r_i are determined recursively by

$$\begin{aligned} p_{i+2} &= (4s^2+2)p_{i+1} - p_i, \qquad & p_1 &= (2s^2+1)p_0 + 2s(s^2+1)r_0, \\ r_{i+2} &= (4s^2+2)r_{i+1} - r_i, \qquad & r_1 &= 2sp_0 + (2s^2+1)r_0. \end{aligned}$$

The crossed ladder solution is

$$(x, y, z, a, b) = (2p_i r_i s,\ p_i(p_i^2 - r_i^2 s),\ p_i^2 - r_i^2 s^2,\ p_i(r_i^2 + s^2),\ p_i^2 + r_i^2 s^2),$$

and the corresponding point on (1.6) is given by:

$$M = p_i, \qquad (X, Y, Z) = (p_i + r_i s,\ p_i - r_i s,\ 2p_i(r_i^2 + s^2)).$$

Taking $(p_0, r_0) = (1, 1)$, then $(p_1, s_1) = (2s^3 + 2s^2 + 2s + 1,\ 2s^2 + 2s + 1)$, and we obtain the sequence p_i, $i \geqslant 1$ as:

$$1 + 2s + 2s^2 + 2s^3,\ 1 + 4s + 8s^2 + 12s^3 + 8s^4 + 8s^5,\ \dots$$

As an example, the former corresponds to crossed ladder solution

$$\begin{aligned}(x, y, z, a, b) =&(2s(1 + 2s + 2s^2)(1 + 2s + 2s^2 + 2s^3),\\ &(1 + s)(1 + 2s)(1 + s + 2s^2)(1 + 2s + 2s^2 + 2s^3),\\ &(1 + s)(1 + 2s)(1 + s + 2s^2),\\ &(1 + 2s + 2s^2 + 2s^3)(1 + 4s + 9s^2 + 8s^3 + 4s^4),\\ &1 + 4s + 9s^2 + 16s^3 + 20s^4 + 16s^5 + 8s^6),\end{aligned}$$

and point on (1.6) with $M = 2s^3 + 2s^2 + 2s + 1$,

$$(X, Y, Z) = ((2s+1)(2s^2+s+1),\ s+1,\ 2(2s^3+2s^2+2s+1)(4s^4+8s^3+9s^2+4s+1)).$$

Numerical values of M that arise from these parametrizations are

$$\boxed{M = 7, 11, 29, 41, 79, 103, 169, 199, 209, \dots}$$

(with $M = 199$ arising from $M = 1 + 4s + 8s^2 + 12s^3 + 8s^4 + 8s^5$, the other values arising from $M = 1 + 2s + 2s^2 + 2s^3$).

If instead we take $(p_0, r_0) = (\pm s^2, s)$, then the resulting sequence p_i is

$$s^2(3 + 4s^2),\ s^2(5 + 20s^2 + 16s^4),\ \dots$$

which is a special case of the sequence derived under Case III, and is not considered further here.

If at (3.6) we consider instead r to be a fixed parameter, then

$$p^2 - s^2(r^2 - 1) = r^2. \tag{3.7}$$

Analogously,

$$p_i + s_i\sqrt{r^2 - 1} = (r + \sqrt{r^2 - 1})^i(p_0 + s_0\sqrt{r^2 - 1}),$$

for an initial solution (p_0, s_0). Taking $(p_0, s_0) = (r, 0)$ we obtain the recurrences

$$\begin{aligned} p_{i+2} &= 2rp_{i+1} - p_i, \qquad & p_0 &= r, \quad & p_1 &= r^2,\\ s_{i+2} &= 2rs_{i+1} - s_i, \qquad & s_0 &= 0, \quad & s_1 &= r,\end{aligned}$$

giving the sequence of p_i (and hence M) as

$$r(2r^2 - 1), \quad r^2(4r^2 - 3), \quad r(8r^4 - 8r^2 + 1), \quad \dots$$

Numerical values of M arising from these parametrizations are:

$$\boxed{M = 14, 51, 52, 124, 194, 245, \dots}$$

3.3. Case III

We suppose $(r,s)=(p,1)$, which demands

$$\frac{q(p^2-1)}{p^2-q^2}\in\mathbb{Z}. \tag{3.8}$$

Solutions are generated by the recurrence relations

$$p_i = np_{i-1} - p_{i-2}, \quad i \geqslant 3, \qquad p_1=-1,\ p_2=1,$$

$$q_i = nq_{i-1} - q_{i-2}, \quad i \geqslant 3, \qquad q_1=1,\ q_2=1,$$

where, on taking $(p,q)=(p_i,q_i)$, we have corresponding value of M equal to $(\frac{n+2}{4})q_i$. Accordingly, we take $n\equiv 2 \bmod 4$. The first three values of (p_i,q_i) with the corresponding M and point (X,Y,Z) on (1.6) are as follows, where we require $n\equiv 2 \bmod 4$.

$$\begin{gathered}(p_3,q_3)=(n+1,n-1), \quad M=\left(\frac{n+2}{4}\right)(n-1),\\ (X,Y,Z)=\left(n,1,(n-1)\left(\frac{1}{2}n^2+n+1\right)\right);\end{gathered} \tag{3.9}$$

$$\begin{gathered}(p_4,q_4)=(n^2-n+1,n^2-n-1), \quad M=\left(\frac{n+2}{4}\right)(n^2-n-1),\\ (X,Y,Z)=\left(n^2-1,n,\frac{1}{2}(n^2-n-1)(n^4+2n^3-n^2-2n+2)\right);\end{gathered}$$

$$\begin{gathered}(p_5,q_5)=(n^3+n^2-2n-1,n^3-n^2-2n+1),\\ M=\left(\frac{n+2}{4}\right)(n^3-n^2-2n+1),\\ (X,Y,Z)=\\ \left(n(n^2-2),n^2-1,\frac{1}{2}(n^3-n^2-2n+1)(n^6+2n^5-3n^4-6n^3+2n^2+4n+2)\right).\end{gathered}$$

Numerical values of M arising from these parametrizations are:

$$\boxed{M=7,10,22,27,41,45,52,58,76,85,115,126,162,175,217,\ldots}$$

The solution at (3.9) has M quadratic in the parameter n, and we can find all the corresponding parametrizations of (1.6) because it may be shown that the curve (1.8) which equals

$$E_n: v^2 = u\left(u-\frac{1}{16}(n-1)^2(n+2)^2\right)\left(u-\frac{1}{16}(n-2)(n+3)(n^2+n+2)\right),$$

is of rank 1 over $\mathbb{Q}(n)$ with generator

$$P(u,v) = \left(\frac{1}{16}(n-2)(n-1)(n^2+n+2),\ \frac{1}{16}n(n-2)(n-1)(n^2+n+2)\right). \tag{3.10}$$

(That the rank is 1 follows from Shioda's formula for the K3 elliptic surface represented by E_n; that P is a generator follows from computing its height of 7/8, and using arguments similar to those of Kuwata [6]). So, for example,

$$2P = \left(\frac{(n-1)^2(n^2+2n+2)^2}{16n^2},\ -\frac{(n-1)^2(n+1)(n^2+1)(n^2+2n+2)}{16n^3}\right),$$

leading to

$$(X,Y,Z) = \Big(-(n+1)(n^2+1),\ n(n^2+2n+2),$$
$$\frac{1}{2}(n^8+4n^7+6n^6+4n^5-n^4-4n^3+2n^2+4n+2)\Big),$$

and

$$3P = \Big(\frac{(n-2)(n-1)(n^2+n+2)(n^4+3n^3+3n^2+n+1)^2}{16(n^4+n^3+n^2-n-1)^2},$$
$$\frac{(n-2)(n-1)n(n^2+n+2)(n^3-n-1)(n^3+2n^2+3n+3)(n^4+3n^3+3n^2+n+1)}{16(n^4+n^3+n^2-n-1)^3}\Big),$$

leading to

$$(X,Y,Z) =$$
$$= \Big(n(n^3-n-1)(n^3+2n^2+3n+3),\ (n^4+3n^3+3n^2+n+1)(n^4+n^3+n^2-n-1),$$
$$-\frac{1}{2}(n-1)(n^2+2n+2)(n^{14}+6n^{13}+17n^{12}+36n^{11}+66n^{10}+104n^9+139n^8$$
$$+140n^7+95n^6+38n^5+4n^4+6n^2+4n+1)\Big).$$

An alternative approach to making (3.8) integral is to set $p=F_n$, $q=F_{n+1}+F_{n-1}$, where n is odd (so that $F_n^2-1=F_{n+1}F_{n-1}$). Then $M=\frac{1}{4}(F_{n+1}+F_{n-1})$, and is integral precisely when $n \equiv 3 \bmod 6$. This gives rise to numerical values for M equal to $M = \boxed{19, 341, \ldots}$. Alternatively, setting $p=F_{n+1}+F_{n-1}$, $q=F_n$, and using that $(F_{n+1}+F_{n-1})^2-1=5F_{n+1}F_{n-1}$ for n odd, then $M=\frac{5}{4}F_n$, which is integral precisely when $n \equiv 0 \bmod 6$. Corresponding numerical values are $M = \boxed{10, 180, 3230, \ldots}$

3.4. Case IV

We demand $p+q=r-s$, $pq=rs$, by setting

$$p=n+1,\ q=n-1,\ r=m+n,\ s=m-n,$$

where

$$m^2 - 2n^2 = -1, \tag{3.11}$$

in which case $M = m$, with corresponding (x, y, z, a, b) given by

$$(n^2 - 1,\ 2mn,\ 2n,\ m^2 + n^2,\ n^2 + 1).$$

The solutions of (3.11) are well known, corresponding to $m + n\sqrt{2}$ being an odd power of the fundamental unit $1 + \sqrt{2}$ in the ring $\mathbb{Z}[\sqrt{2}]$: namely $m = p_i$, where $p_i = 6p_{i-1} - p_{i-2}$, $i \geqslant 2$, and $p_0 = 1$, $p_1 = 7$. This gives numerical values $M = \boxed{7, 41, 239, \ldots}$

3.5. Case V

We demand $p^2 - q^2 = r^2 - s^2$ and put $p + q = K(r - s)$, $K(p - q) = r + s$. Eliminating r, s,

$$M(K^4 - 1)p^2 - 2(M + 2K^2 + MK^4)pq + M(K^4 - 1)q^2 = 0$$

whose discriminant being square demands that

$$(M + K^2)(MK^2 + 1) = \square.$$

Assuming $M + K^2 = (A + K)^2$, then $M = 2AK + A^2$, so that $2AK^3 + A^2K^2 + 1 = \square = (A\rho - 1)^2$, say, giving $A = 2(\rho + K^3)/(\rho^2 - K^2)$. If we choose $\rho = K^2$ there results $K = (A + 2)/A$, $M = A^2 + 2A + 4$, so that setting $w = A + 1$, we have

$$M = w^2 + 3, \qquad (p, q, r, s) = (w(3 + w^2),\ -2(1 + w^2),\ -w(1 + w^2),\ 2),$$

with point on (1.6)

$$(X, Y, Z) = \Big((-1+w)(2-w+w^2),\ (1+w)(2+w+w^2),\ 2(3+w^2)(4+w^2+2w^4+w^6)\Big).$$

Numerical values are $\boxed{M = 12, 19, 28, 39, 52, 67, 84, 103, 124, 147, 172, 199, 228, \ldots}$

Choosing instead $\rho = K + 2$, then

$$\begin{aligned} A &= (K^2 - K + 2)/2, \quad M = (K + 1)(K + 2)(K^2 - K + 2)/4, \\ (p, q, r, s) &= \Big((K + 1)^2(K^2 - K + 2),\ (K - 1)(K + 2)(K^2 + 1), \\ &\qquad 2(K + 1)(K^2 + 1),\ 2(K - 1)\Big), \end{aligned}$$

with point on (1.6)

$$\begin{aligned} (X, Y, Z) = \Big(&2K(K^3 + K^2 + 2),\ 2(K^2 + K + 2), \\ &2(K + 1)(K + 2)(K^2 - K + 2)(K^6 + 2K^5 + 3K^4 + 4K^3 + 4K^2 + 2)\Big). \end{aligned}$$

Numerical values are $\boxed{M = 7, 12, 33, 40, 96, 105, 220, \ldots}$

Finally, if we set $2AK^3 + A^2K^2 + 1 = (AK + K^2 - 1)^2$, then $A = K(K^2 - 2)/2$ and $M = K^2(K^4 - 4)/4$, with corresponding $(X, Y, Z) = (2(K^4 - 2), 2K^2, 2(K^2 - 2)(K^2 + 2)(K^8 - 2K^4 + 2))$. On setting $K^2 = 2w$ this gives rise to the parametrization

$$M = 2w(w^2 - 1), \quad (X, Y, Z) = (2w^2 - 1, w, (w^2 - 1)(8w^4 - 4w^2 + 1)).$$

Numerical values are $\boxed{M = 12, 48, 120, 240, \ldots}$.

The curve with $M = w^2 + 3$ represents a K3 elliptic surface:

$$E : v^2 = u\left(u - (w^2 + 3)^2\right)\left(u - (w^2 + 4)(w^2 + 2)\right).$$

and it is possible to show by the Shioda formula that the rank of E over $\mathbb{C}(w)$ is equal to 1. It is likely that

$$P = \left(\frac{(w^2 - w + 2)^2(w^2 + 3)}{(w - 1)^2}, \frac{(w + 1)(w^2 + 3)(w^2 + w + 2)(w^2 - w + 2)}{(w - 1)^3}\right)$$

is a generator for the group (in which case the field of definition of the group is actually $\mathbb{Q}(w)$), and this could be verified as above using a height argument, although we have not undertaken the computation. As before, therefore, we can determine infinitely many parametrized solutions to the ladder problem by computing multiples of P.

4. Rank data

We list here the rank of the elliptic curve (1.8) in the range $3 \leqslant M < 200$ (computed with the aid of Magma [1]).

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0			0	0	0	0	0	1	0	0	1	1	1	0	1	0	0	1	0	1
20	0	0	1	1	0	0	0	1	1	1	1	0	0	1	0	0	0	0	1	1
40	1	2	1	0	1	1	0	1	1	0	0	1	2	1	1	0	0	1	1	2
60	0	1	0	0	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	1
80	1	1	1	1	1	1	0	0	1	0	0	0	1	1	0	0	1	1	0	0
100	1	0	1	3	0	1	0	1	1	1	0	1	1	1	0	2	0	0	1	1
120	1	1	0	0	2	0	1	1	0	0	0	2	1	1	0	1	1	2	1	1
140	1	0	0	0	0	1	1	1	1	0	1	0	0	0	0	0	1	1	1	1
160	0	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1	0	1	1	2
180	2	1	1	1	0	0	1	1	1	0	0	0	0	0	2	1	1	0	1	2

There are 111 instances in the range $1 < M < 200$ of curves with positive rank, of which we have identified 39 as coming from parametrized families. Of course it is unlikely that every curve of positive rank arises from a parametrization. For example, the curve at $M = 127$ has rank 1 and the smallest solution of the equation at (1.6) is given by $(X, Y, Z) = (59914079, 205805825, 313222918714897363 4)$.

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