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A common fixed point theorem via a generalized contractive condition

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Abstract

We prove a common fixed point theorem for mappings satisfying a generalized contractive condition which generalizes the results of [3, 4, 12, 15, 19, 20, 24] and we correct the errors of [7, 12, 20].

Keywords: Metric space, weakly compatible mappings, common fixed point.

MSC: 47H10, 54H25

1. Introduction

Sessa [21] defined S and T to be weakly commuting as a generalization of commuting if for all $x \in X$.

$$d\left(STx, TSx\right) \leqslant d\left(Tx, Sx\right).$$

Jungck [9] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d\left(STx_n, TSx_n\right) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [9, 21]. Jungck et al [10] defined S and T to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d\left(STx_n, T^2x_n\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\left(TSx_n, S^2x_n\right) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Example are given to show that the two concepts of compatibility are independent, see [10]. Recently, Pathak and Khan [16] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\left(TSx_n, S^2x_n\right) \leqslant \frac{1}{2}\left[\lim_{n\to\infty} d\left(TSx_n, Tt\right) + \lim_{n\to\infty} d\left(Tt, T^2x_n\right)\right],$$
$$\lim_{n\to\infty} d\left(STx_n, T^2x_n\right) \leqslant \frac{1}{2}\left[\lim_{n\to\infty} d\left(STx_n, St\right) + \lim_{n\to\infty} d\left(St, S^2x_n\right)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Clearly compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [16]. However, compatible mappings of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [16]. Pathak et al [17] defined S and T to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d\left(S^2x_n, T^2x_n\right) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous, see [17]. Pathak et al [18] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\left(TSx_n, S^2x_n\right) \leqslant \frac{1}{3}\left[\lim_{n\to\infty} d\left(TSx_n, Tt\right) + \lim_{n\to\infty} d\left(Tt, S^2x_n\right) + \lim_{n\to\infty} d\left(Tt, T^2x_n\right)\right],$$
$$\lim_{n\to\infty} d\left(STx_n, T^2x_n\right) \leqslant \frac{1}{3}\left[\lim_{n\to\infty} d\left(STx_n, St\right) + \lim_{n\to\infty} d\left(St, T^2x_n\right) + \lim_{n\to\infty} d\left(St, S^2x_n\right)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous, see [18]. Pant [15] defined S and T to be reciprocally continuous if

$$\lim_{n\to\infty} STx_n = St \text{ and } \lim_{n\to\infty} TSx_n = Tt,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. It is clear that if S and T are both continuous, then they are reciprocally continuous, but the converse is not true. Moreover, it was proved in [15] that in the setting of common fixed point theorem for compatible mappings satisfying contractive conditions, the continuity of one of the mappings S and T implies their reciprocal continuity, but not conversely.

2. Preliminaries

Definition 2.1 (See [11]). S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Lemma 2.2 (See [9, 10, 16, 17, 18]). *If S and T are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.*

The converse is not true in general, see [4].

Definition 2.3 (See [13]). S and T are said to be $R-$weakly commuting if there exists an $R > 0$ such that

$$d\,(STx, TSx) \leqslant Rd\,(Tx, Sx) \ \text{ for all } x \in X. \tag{2.1}$$

Definition 2.4 (See [14]). S and T are pointwise $R-$weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [14] that R-weakly commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Lemma 2.5 (See [22]). *For any $t \in (0, \infty)$, $\psi(t) < t$ iff $\lim_{n\to\infty} \psi^n(t) = 0$, where ψ^n denotes the n-times repeated composition of ψ with itself.*

Several authors proved fixed point and common fixed point theorem for mappings satisfying contractive conditions of integral type, see [1, 3, 4, 5, 6, 7, 12, 19, 20]. The following theorem was proved by [3].

Theorem 2.6 (See [3]). *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying*

$$S\,(X) \subset B\,(X) \quad \textit{and} \quad T(X) \subset A\,(X)\,,$$

$$\int_0^{d(Sx,Ty)} \varphi(t)\,\mathrm{d}t \leqslant \psi\left(\int_0^{M(x,y)} \varphi(t)\,\mathrm{d}t\right)$$

for all $x, y \in X$, $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a right continuous function such that $\psi(0) = 0$ and $\psi(s) < s$ for all $s > 0$ and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$is a Lebesgue integrable mapping which is summable and satisfies

$$\int_0^{\epsilon} \varphi(t)\,\mathrm{d}t > 0,$$

$$M(x,y) = \max\left\{ d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{1}{2}[d(Sx,By) + d(Ty,Ax)] \right\}.$$

If one of $A\,(X)\,, B\,(X)\,, S(X)$ and $T\,(X)$ is a complete subspace of X, then A and S have a coincidence point and B and T have a coincidence point. Further, if S and A as well as T and B are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Recently, Zhang [24] and Aliouche [2] proved common fixed point theorems using generalized contractive conditions in metric spaces.

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$ and $F\colon R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,
(ii) F is nondecreasing on R_A^+,
(iii) F is continuous.
Define $\mathcal{F}[0, A) = \{F : F \text{ satisfies (i)–(iii)}\}$.

Lemma 2.7 (See [24]). *Let $A \in (0, \infty]$, $F \in \mathcal{F}[0, A)$. If $\lim_{n\to\infty} F(\epsilon_n) = 0$ for $\epsilon_n \in R_A^+$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

The following examples were given in [24].
(i) Let $F(t) = t$, then $F \in \mathcal{F}[0, A)$ for each $A \in (0, \infty]$.
(ii) Suppose that φ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^\epsilon \varphi(t)\,\mathrm{d}t > 0 \text{ for each } \epsilon \in (0, A).$$

Let $F(t) = \int_0^t \varphi(s)\,\mathrm{d}s$, then $F \in [0, A)$.
(iii) Suppose that ψ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^\epsilon \psi(t)\,\mathrm{d}t > 0 \text{ for each } \epsilon \in (0, A)$$

and φ is nonnegative, Lebesgue integrable on $\left[0, \int_0^A \psi(s)\,\mathrm{d}s\right)$ and satisfies

$$\int_0^\epsilon \varphi(t)\,\mathrm{d}t > 0 \text{ for each } 0 < \epsilon < \int_0^A \psi(s)\,\mathrm{d}s.$$

Let $F(t) = \int_0^{\int_0^t \psi(s)\,\mathrm{d}s} \varphi(u)\,\mathrm{d}u$, then $F \in \mathcal{F}[0, A)$.
(iv) If $G \in [0, A)$ and $F \in \mathcal{F}[0, G(A - 0))$, then a composition mapping $F \circ G \in \mathcal{F}[0, A)$. For instance, let $H(t) = \int_0^{F(t)} \varphi(s)\,\mathrm{d}s$, then $H \in \mathcal{F}[0, A)$ whenever $F \in \mathcal{F}[0, A)$ and φ is nonnegative, Lebesgue integrable on $\mathcal{F}[0, F(A - 0))$ and satisfies

$$\int_0^\epsilon \varphi(t)\,\mathrm{d}t > 0 \text{ for each } \epsilon \in (0, F(A - 0)).$$

Let $A \in (0, \infty]$ and $\psi\colon R_A^+ \to \mathbb{R}_+$ satisfying
(i) $\psi(t) < t$ for all $t \in (0, A)$
(ii) ψ is upper semi-continuous.
(iii) ψ is nondecreasing on R_A^+,
Define $\Psi[0, A) = \{\psi : \psi \text{ satisfies (i)-(iii)}\}$.

3. Main results

Theorem 3.1. *Let (X, d) be a metric space and $D = \sup\{d(x, y) : x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. Let A_1, A_2, S and T be self-mappings of (X, d) satisfying*

$$A_1(X) \subset T(X) \text{ and } A_2(X) \subset S(X),$$

$$F(d\,(A_1x, A_2y)) \leqslant \psi(F(L(x,y))) \tag{3.1}$$

for all x, y in X, where

$$L(x,y) = \max\Big\{d(Sx,Ty), d(A_1x,Sx), d(Ty,A_2y), \frac{1}{2}[d(Sx,A_2y)+d(A_1x,Ty)]\Big\},$$

$F \in \mathcal{F}[0,A)$ and $\psi \in \Psi[0,F(A-0))$ for all $A \in (0,\infty]$. Suppose that the pair (A_1,S) is weakly compatible and there exists $w \in C(A_2,T)$: the set of coincidence points of A_2 and T such that $A_2Tw = TA_2w$. If one of $A_1(X), A_2(X), S(X)$ and $T(X)$ is a complete subspace of X, then A_1, A_2, S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary point in X. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = A_1x_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+2} = A_2x_{2n+1}$$

for all $n = 0,1,2,\ldots$. As in the proof of [2], $\{y_n\}$ is a Cauchy sequence in X. Assume that $S(X)$ is complete. Therefore

$$\lim_{n\to\infty} A_1x_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} A_2x_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = z = Su$$

for some $u \in X$. If $A_1u \neq z$ using (3.1) we obtain

$$F(d\,(A_1u, A_2x_{2n+1})) \leqslant \psi(F(L(u,x_{2n})))$$

where

$$L(u,x_{2n}) = \max\Big\{d(Su,Tx_{2n+1}), d(A_1u,Su), d(Tx_{2n+1},A_2x_{2n+1}), \frac{1}{2}[d(Su,A_2x_{2n+1})+d(A_1u,Tx_{2n+1})]\Big\}.$$

Letting $n \to \infty$, we get

$$F(d\,(A_1u,z)) \leqslant \psi(F(d\,(A_1u,z))) < F(d\,(A_1u,z))$$

which is a contradiction and so $z = A_1u = Su$. If $z \neq A_2w$, applying (3.1) we obtain

$$F(d\,(A_1u, A_2w)) \leqslant \psi(F(d\,(A_1u,A_2w)))$$

where

$$L(u,v) = \max\Big\{d(Su,Tw), d(A_1u,Su), d(Tw,A_2w), \frac{1}{2}[d(Su,A_2w)+d(A_1u,Tw)]\Big\}.$$

Hence

$$F(d\,(z,A_2w)) \leqslant \psi(F(d\,(z,A_2w))) < F(d\,(z,A_2w)).$$

which is a contradiction and so $z = A_1u = Su = A_2w = Tw$.

Since the pairs (A_1, S) is weakly compatible and there exists $w \in C(A_2, T)$ such that $A_2Tw = TA_2w$, we have $Sz = A_1z$ and $Tz = A_2z$.

If $A_1z \neq z$ we have by (3.1)

$$F(d\,(A_1z, A_2w)) \leqslant \psi(F(L(z,w)))$$

where

$$L(z,w) = \max\Big\{d(Sz,Tw), d(A_1z,Sz), d(Bw,A_2w), \frac{1}{2}[d(Sz,A_2w) + d(A_1z,Tw)]\Big\}.$$

Therefore

$$F(d\,(A_1z, z)) \leqslant \psi(F(d\,(A_1z, z))) < F(d\,(A_1z, z))$$

and so $A_1z = Sz = z$. Similarly, we can prove that $A_2z = Tz = z$.

The proof is similar when $T\,(X)$ is assumed to be a complete subspace of X. The case in which $A_1\,(X)$ or $A_2\,(X)$ is a complete subspace of X is similar to the case in which $T\,(X)$ or $S\,(X)$ respectively is complete since $A_1\,(X) \subset T\,(X)$ and $A_2\,(X) \subset S\,(X)$. The uniqueness of z follows from (3.1). □

Theorem 3.1 generalizes Theorem 2.6 of [3].

Corollary 3.2. *Let (X,d) be a metric space and $D = \sup\{d(x,y) : x,y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. Let $\{A_i\}$, $i = 1,2,\ldots,S$ and T be self-mappings of (X,d) satisfying*

$$A_1(X) \subset T(X) \text{ and } A_i(X) \subset S(X),\ i \geqslant 2$$

and

$$F(d\,(A_1x, A_iy)) \leqslant \psi(F(L_i(x,y))),\ i \geqslant 2$$

for all x,y in X, where

$$L_i(x,y) = \max\Big\{d(Sx,Ty), d(A_1x,Sx), d(A_iy,Ty), \frac{1}{2}[d(Sx,A_iy) + d(A_1x,Ty)]\Big\},$$

$F \in \digamma[0,A)$ and $\psi \in \Psi[0,F(A-0))$ for all $A \in (0,\infty]$. Suppose that the pair (A_1,S) is weakly compatible and there exists $w \in C(A_i,T)$: the set of coincidence points of A_i and T such that $A_iTw = TA_iw$ for some $i \geqslant 2$. If one of $A_i\,(X), S\,(X)$ and $T\,(X)$ is a complete subspace of X. Then A_i, S and T have a unique common fixed point in X.

If $\varphi(t) = 1$ in Corollary 3.2, we get a generalization of a theorem of [15]. The following example illustrates our corollary 3.2.

Example 3.3. Let $X = [0,10]$ be endowed with the metric $d\,(x,y) = |x-y|$,

$$Sx = \begin{cases} 0, & \text{if } x = 0, \\ x+8, & \text{if } x \in (0,2]\,, \\ x-2, & \text{if } x \in (2,10]\,, \end{cases} \qquad Tx = \begin{cases} 0, & \text{if } x = 0, \\ x+5, & \text{if } x \in (0,2]\,, \\ x-2, & \text{if } x \in (2,10]\,, \end{cases}$$

$$A_1x = \begin{cases} 3, & \text{if } x \in (0,2]\,, \\ 0, & \text{if } x \in \{0\} \cup (2,10]\,, \end{cases} \qquad A_2x = \begin{cases} 0, & \text{if } x \in [0,2]\,, \\ 4, & \text{if } x \in (2,10]\,, \end{cases}$$

$$A_3x = \begin{cases} 0, & \text{if } x \in [0,2]\,, \\ 5, & \text{if } x \in (2,10]\,, \end{cases} \qquad A_4x = \begin{cases} 0, & \text{if } x \in [0,2]\,, \\ 6, & \text{if } x \in (2,10]\,, \end{cases}$$

$$A_ix = \begin{cases} 2 + \frac{2}{i}, & \text{if } x \in (0,2]\,, \\ 0, & \text{if } x \in \{0\} \cup (2,10]\,, \end{cases} \qquad \text{for all } i > 4.$$

The pair (A_1, S) is weakly compatible, but it is not compatible of type (A), (B), (P) and (C), see [6].

$A_1 (X) \subset T(X)$ and $A_i (X) \subset S(X)$.

The pair (A_i, T), $i > 4$, is weakly compatible because A_i and T commute at their coincidence point $x = 0$, but it is not compatible of type (A), (B), (P) and (C).

Let $x_n = 2 + \frac{1}{n}$. We have $Tx_n = \frac{1}{n}$ and $A_ix_n = 0$, hence

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} A_ix_n = t = 0.$$

In the other hand, $A_iTx_n = A_i(\frac{1}{n}) = 2 + \frac{2}{i}$ and $TA_ix_n = T0 = 0$ and so $\lim_{n\to\infty} d(A_iTx_n, TA_ix_n) = 2 + \frac{2}{i} \neq 0$. Therefore, the pair (A_i, T) is not compatible.

$A_i^2x_n = A_i0 = 0$ and $T^2x_n = T\left(\frac{1}{n}\right) = 5 + \frac{1}{n}$, so $\lim_{n\to\infty} \left|TA_ix_n - A_i^2x_n\right| = 0$ and $\lim_{n\to\infty} \left|A_iTx_n - T^2x_n\right| = \lim_{n\to\infty}(3 + \frac{1}{n} - \frac{2}{i}) \neq 0$ for all $i > 3$. Then, (A_i, T) is not compatible of type (A).

$$\begin{aligned} \lim_{n\to\infty} \left|A_iTx_n - T^2x_n\right| = 3 - \frac{2}{i} &> \frac{1}{2}\left[\lim_{n\to\infty} |A_iTx_n - A_i0| + \lim_{n\to\infty} \left|A_i0 \ - A_i^2x_n\right|\right] \\ &= \frac{1}{2}\left|2 + \frac{2}{i}\right| = \frac{1}{i} + 1, \end{aligned}$$

hence (A_i, T) is not compatible of type (B).

$\lim_{n\to\infty} \left|A_i^2x_n - T^2x_n\right| = \lim_{n\to\infty}(5 + \frac{1}{n}) = 5 \neq 0$. Therefore, (A_i, T) is not compatible of type (P).

$$\begin{aligned} &\lim_{n\to\infty} \left|A_iTx_n - T^2x_n\right| = 3 - \frac{2}{i} \\ &> \frac{1}{3}\left[\lim_{n\to\infty} |A_iTx_n - A_i0| + \lim_{n\to\infty} \left|A_i0 - T^2x_n\right| + \lim_{n\to\infty} \left|A_i0 \ - A_i^2x_n\right|\right] \\ &= \frac{1}{3}\left(7 + \frac{2}{i}\right) \end{aligned}$$

for $i > 4$. So, the pair (A_i, T) is not compatible of type (C).

It can be verified that the pairs (A_2, T) , (A_3, T) and (A_4, T) are not weakly compatible because $x = 6$ is a coincidence point of A_2 and T, but $A_2T6 = 4 \neq$

$TA_26 = 2$, $x = 7$ is a coincidence point of A_3 and T, but $A_3T(7) = 5 \neq TA_3(7) = 3$ and $x = 8$ is a point of coincidence for A_4 and T, but $A_4T(8) = 6 \neq TA_4(8) = 4$.

Now, we begin to verify the rest of conditions of Corollary 3.2. Let $F(t) = \ln(1+t)$ and $\psi(t) = ht$, where $0 \leqslant h < 1$ and $t > 0$. Set

$$R = \ln(1+|A_1x - A_iy|) - h\max\left\{\begin{array}{l} \ln(1+|Sx-Ty|), \ln(1+|A_1x-Sx|), \\ \ln(1+|A_iy-Ty|), \\ \frac{1}{2}\left[\ln(1+|A_1x-Ty|)+\ln(1+|Sx-A_iy|)\right] \end{array}\right\}$$

We have the following cases. If $x = 0$ and $y = 0$ we get $R \leqslant 0$ for all $0 \leqslant h < 1$. If $x = 0$ and $y \in (0, 2]$, we get

$$R = \ln\left(3+\frac{2}{i}\right) - h\max\left\{\begin{array}{l} \ln(y+6), \ln\left(y+4-\frac{2}{i}\right), \\ \frac{1}{2}\left[\ln(y+6)+\ln\left(3+\frac{2}{i}\right)\right] \end{array}\right\} \leqslant 0$$

for $h \geqslant \frac{\ln\left(3+\frac{2}{i}\right)}{3\ln 2}$ and so there exists $0 \leqslant h < 1$. If $x = 0$ and $y \in (2, 10]$, we get

$$R = -h\max\left\{\ln(y-1), \ln(y-1), \ln(y-1)\right\} \leqslant 0$$

for all $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y = 0$, we get

$$R = \ln 4 - h\max\left\{\begin{array}{l} \ln(x+9), \ln(x+6), \\ \frac{1}{2}\left[\ln 4 + \ln(x+9)\right] \end{array}\right\} \leqslant 0$$

for $h \geqslant \frac{\ln 4}{\ln 11}$ and so there exists $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y \in (0, 2]$, we get

$$R = \ln\left(2-\frac{2}{i}\right) - h\max\left\{\begin{array}{l} \ln(x-y+4), \ln(x+6), \ln\left(y+4-\frac{2}{i}\right), \\ \frac{1}{2}\left[\ln(y+3)+\ln\left(x+7-\frac{2}{i}\right)\right] \end{array}\right\} \leqslant 0$$

for $h \geqslant \frac{\ln\left(3-\frac{2}{i}\right)}{\ln 8}$. Hence, there exists $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y \in (2, 10]$, we get

$$R = \ln 4 - h\max\left\{\begin{array}{l} \ln(x+11-y), \ln(x+6), \ln(y-1), \\ \frac{1}{2}\left[\ln(|5-y|+1)+\ln(x+9)\right] \end{array}\right\} \leqslant 0$$

for $h \geqslant \frac{\ln 4}{\ln 11}$. Hence, there exists $0 \leqslant h < 1$. If $x \in (2, 10]$ and $y = 0$, we get

$$R = -h\max\left\{\ln(x-1), \ln(x-1), 0, \frac{1}{2}\ln(x-1)\right\} \leqslant 0$$

for all $h \geqslant 0$. Hence, there exists $0 \leqslant h < 1$. If $x \in (2, 10]$ and $y \in (0, 2]$, we get

$$R = \ln\left(3+\frac{2}{i}\right) - h\max\left\{\begin{array}{l} \ln(|x-(y+7)|+1), \ln(x-1), \\ \ln\left(y+3-\frac{2}{i}\right), \frac{1}{2}\left[\ln(y+5)+\ln\left(\left|x-4-\frac{2}{i}\right|+1\right)\right] \end{array}\right\} \leqslant 0$$

for $h \geqslant \frac{\ln\left(3+\frac{2}{i}\right)}{\ln 9}$. Hence, there exists $0 \leqslant h < 1$. If $x, y \in (2, 10]$ we get

$$R = -h\max\left\{\begin{array}{l} \ln(|x-y|+1), \ln(x-1), \ln(y-1), \\ \frac{1}{2}\left[\ln(y-1)+\ln(x-1)\right] \end{array}\right\} \leqslant 0$$

for all $0 \leqslant h < 1$.

Now, we verify that (A_2, T) and (A_3, T) satisfy all the conditions of Theorem 4.2. Set

$$R_1 = \int_0^{|A_1x - A_2y|} \frac{1}{1+t}\,\mathrm{d}t - \\ - h\max\left\{ \begin{array}{l} \int_0^{|Sx-Ty|} \frac{1}{1+t}\,\mathrm{d}t, \int_0^{|A_1x-Sx|} \frac{1}{1+t}\,\mathrm{d}t, \int_0^{|A_2y-Ty|} \frac{1}{1+t}\,\mathrm{d}t, \\ \frac{1}{2}\left[\int_0^{|A_1x-Ty|} \frac{1}{1+t}\,\mathrm{d}t + \int_0^{|Sx-A_2y|} \frac{1}{1+t}\,\mathrm{d}t\right] \end{array} \right\}$$

We have the following cases. If $x = 0$ and $y = 0$ we get $R_1 \leqslant 0$ for all $0 \leqslant h < 1$. If $x = 0$ and $y \in (0, 2]$, we get

$$R_1 = -h\max\left\{\ln(y+6), 0, \ln(y+6), \frac{1}{2}[y+6]\right\} \leqslant 0$$

for all $0 \leqslant h < 1$. If $x = 0$ and $y \in (2, 10]$, we get

$$R_1 = \ln 5 - h\max\left\{ \begin{array}{l} \ln(y-1), \ln(|y-6|+1), \\ \frac{1}{2}[\ln(y-1) + \ln 5] \end{array} \right\} \leqslant 0$$

for $h \geqslant \frac{\ln 5}{\ln 9}$, hence there exists $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y = 0$, we get

$$R_1 = \ln 4 - h\max\left\{ \begin{array}{l} \ln(x+9), \ln(x+6), 0, \\ \frac{1}{2}[\ln 4 + \ln(x+9)] \end{array} \right\} \leqslant 0$$

for all $h \geqslant \frac{\ln 4}{\ln 11}$. Hence, there exists $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y \in (0, 2]$, we get

$$R_1 = \ln 4 - h\max\left\{ \begin{array}{l} \ln(4+x-y), \ln(x+6), \ln(y+6), \\ \frac{1}{2}[\ln(y+3) + \ln(x+9)] \end{array} \right\} \leqslant 0$$

for $h \geqslant \frac{\ln 4}{\ln 8}$. Hence there exists $0 \leqslant h < 1$. If $x \in (0, 2]$ and $y \in (2, 10]$, we get

$$R_1 = \ln 2 - h\max\left\{ \begin{array}{l} \ln 11, \ln(x+6), \ln(|y-6|+1), \\ \frac{1}{2}[\ln(|5-y|+1) + \ln(x+5)] \end{array} \right\} \leqslant 0$$

for $h \geqslant \frac{\ln 2}{\ln 11}$. Hence,there exists $0 \leqslant h < 1$. If $x \in (2, 10]$ and $y = 0$, we get

$$R_1 = -h\max\left\{\ln(x-1), \ln(x-1), \frac{1}{2}\ln(x-1)\right\} \leqslant 0$$

for all $0 \leqslant h < 1$. In the same manner, if $x \in (2, 10]$ and $y \in (0, 2]$, we get $R_1 \leqslant 0$ for all $0 \leqslant h < 1$. If $x \in (2, 10]$ and $y \in (2, 10]$, we get

$$R_1 = \ln 5 - h\max\left\{ \begin{array}{l} \ln(|x-y|+1), \ln(x-1), \ln(|y-6|+1), \\ \frac{1}{2}[\ln(y-1) + \ln|x-6|+1] \end{array} \right\} \leqslant 0$$

for $h \geqslant \frac{\ln 5}{\ln 9}$. Hence, there exists $0 \leqslant h < 1$.Similarly, we can prove the conditions of Theorem 4.2 if we take the mapping A_3 instead of A_2. Finally we remark that all conditions of our theorem are verified and 0 is the unique common fixed point of A_i, S and T.

The following example support our Theorem 3.1.

Remark 3.4. In this example, Theorem 2.6 of [3] is not applicable since the pair (A_2, T) is not weakly compatible, but Theorem 3.1 is applicable. Also, a theorem of [15] for $A_i = A_2$ for all $i \geqslant 2$ is not applicable since the pairs (A_1, S) and (A_2, T) are not compatible. In the same manner, Theorem 1 of [12] is not applicable.

Remark 3.5. In the proof of Lemma 1 of [20] and Theorem 2.1 of [7], the authors applied the inequality

$$a \leqslant b + c \Longrightarrow \int_0^a \varphi(t)dt \leqslant \int_0^b \varphi(t)dt + \int_0^c \varphi(t)dt$$

which is false in general as it is shown by the following example.

Example 3.6. Let $\varphi(t) = t$, $a = 1$, $b = \frac{1}{2}$ and $c = \frac{3}{4}$. Then $1 < \frac{1}{2} + \frac{3}{4}$, but

$$\begin{aligned}\int_0^1 \varphi(t)dt = \frac{1}{2} &> \int_0^{\frac{1}{2}} \varphi(t)dt + \int_0^{\frac{3}{4}} \varphi(t)dt \\ &= \frac{1}{8} + \frac{9}{32} = \frac{13}{32}.\end{aligned}$$

To correct these errors, the authors should follow the proof of Theorem 2 of [19].

Remark 3.7. In the proof of Theorem 1 of [12], the authors applied the inequality

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \Longrightarrow \{x_n\} \text{ is a Cauchy sequence}$$

which is false in general. It suffices to take $x_n = \frac{1}{n}$, $n \in \mathbb{N}^*$. Thus, To correct this error, the authors should follow the proof of Theorem 2 of [19].

References

[1] Aliouche, A., A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, *J. Math. Anal. Appl.*, 322 (2) (2006), 796–802.

[2] Aliouche, A., Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions, *J. Math. Anal. Appl.*, 341 (1) (2008), 707–719.

[3] Altun, I., Turkoglu, D., Rhoades, B.E., Fixed points of weakly compatible mappings satisfying a general contractive condition of integral type, *Fixed Point Theory And Applications*, Volume 2007 (2007), Article ID 17301, 9 pages.

[4] Branciari, A., A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, 29 (2002), 531–536.

[5] Djoudi, A., Aliouche, A., Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, *J. Math. Anal. Appl.*, 329 (1) (2007), 31–45.

[6] Djoudi, A., Merghadi, F., Common fixed point theorems for maps under a contractive condition of integral type, *J. Math. Anal. Appl.*, 341 (2) (2008), 953–960.

[7] Gairola, U.C., Rawat, A.S., A fixed point theorem for integral type inequality, *Int. Journal of Math. Analysis*, Vol. 2, 2008, no. 15, 709–712.

[8] Bouhadjera, H., Djoudi, A., Common fixed point theorems for pairs of single and multivalued D-maps satisfying an integral type, *Annales Math. et Inf.*, 35, (2008), 43–59.

[9] Jungck, G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, 9 (1986), 771–779.

[10] Jungck, G., Murthy, P.P., Cho, Y.J., Compatible mappings of type (A) and common fixed points, *Math. Japonica.*, 38 (2) (1993), 381–390.

[11] Jungck, G., Common fixed points for non-continuous non-self maps on non metric spaces, *Far East J. Math. Sci.*, 4 (2) (1996), 199–215.

[12] Kohli, J.K., Vashistha, S., Common fixed point theorems for compatible and weakly compatible mappings satisfying general contractive type conditions, *Studii şi Cercetări Ştiinţifice, Seria Matematică*, Universitatea din Bacău, 16 (2006), 33–42

[13] Pant, R.P., Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.*, 188 (1994), 436–440.

[14] Pant, R.P., Common fixed points for four mappings, *Bull. Calcutta. Math. Soc.*, 9 (1998), 281–286.

[15] Pant, R.P., A Common fixed point theorem under a new condition, *Indian J. Pure. Appl. Math.*, 30 (2) (1999), 147–152.

[16] Pathak, H.K., Khan, M.S., Compatible mappings of type (B) and common fixed point theorems of Gregus type, *Czechoslovak Math. J.*, 45 (120) (1995), 685-698.

[17] Pathak, H.K., Cho, Y.J., Kang, S.M., Lee, B.S., Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, *Le Matematiche*, 1 (1995), 15–33.

[18] Pathak, H.K., Cho, Y.J., Khan, S.M., Madharia, B., Compatible mappings of type (C) and common fixed point theorems of Gregus type, *Demonstratio Math.*, 31 (3) (1998), 499–518.

[19] Rhoades, B.E., Two fixed point theorems for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, 63 (2003), 4007–4013.

[20] Vijayaraju, P., Rhoades, B.E., Mohanraj, R., A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, 15 (2005), 2359–2364.

[21] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. Beograd.*, 32 (46) (1982), 149–153.

[22] Singh, S.P., Meade, B.A., On common fixed point theorems, *Bull. Austral. Math. Soc.*, 16 (1977), 49–53.

[23] Suzuki, T., Meir-Keeler contractions of integral type are still Meir-Keeler contractions, *Int. J. Math. Math. Sci.*, 2007, Article ID 39281, 6 pages, 2007. doi:10.1155/2007/39281.

[24] Zhang, X., common fixed point theorems for some new generalized contractive type mappings, *J. Math. Anal. Appl.*, 333 (2) (2007), 780–786.

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