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Generalization of some inequalities for the q-gamma function

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Abstract

In this paper is obtained q-analogue of a double inequality involving the Euler's gamma function proved in [5]. In the same way, the paper [5] generalized papers [1]–[4], this paper will generalize some inequalities for the q-gamma function such as those presented in [9, 10].

Keywords: q-gamma function, Inequalities

MSC: 33D05

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

The Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0.$$

The q-analogue of the gamma function is defined by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^{x+i}}, \quad q \in (0,1). \tag{1.1}$$

The q-psi function is defined as

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x). \tag{1.2}$$

We will make use of the following well known facts

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x), \ \lim_{q\to 1^-} \psi_q(x) = \psi(x). \tag{1.3}$$

R. Askey, [8] derived some properties of the q-gamma function.

Papers [1, 2, 3, 4] were related to some double inequalities involving the gamma function.

In [5] the following theorem is proved:

Theorem 1.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f}, \quad x \geqslant 0, \tag{1.4}$$

where a, b, c, d, e, f are real numbers such that: $a+bx > 0, d+ex > 0, a+bx \leqslant d+ex$.

In both situations:

i) *Let $ef \geqslant bc > 0$. If $\psi(a+bx) > 0$ or $\psi(d+ex) > 0$*

ii) *Let $bc \geqslant ef > 0$. If $\psi(d+ex) < 0$ or $\psi(a+bx) < 0$*

the function f is decreasing for $x \geqslant 0$ and for $x \in [0,1]$ the following double inequality holds:

$$\frac{\Gamma(a+b)^c}{\Gamma(d+e)^f} \leqslant \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f} \leqslant \frac{\Gamma(a)^c}{\Gamma(d)^f}. \tag{1.5}$$

which represents a generalization of inequalities given in [1, 2, 3, 4].

Some of those inequalities were generalized using q-gamma analogue function. Thus T. Kim and C. Adiga [9] proved:

Theorem 1.2. *If $0 < q < 1, a \geqslant 1$ and $x \in [0,1]$ then*

$$\frac{1}{\Gamma_q(1+a)} \leqslant \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)} \leqslant 1. \tag{1.6}$$

Letting q tend to 1 and $a = n$, one obtains q-gamma analogue to the inequality given in [1]. Letting q tend to 1, one obtains q-gamma analogue to the inequality given in [2].

Recently, T. Mansour [10] proved:

Theorem 1.3. *Let $x \in [0,1]$, $q \in (0,1)$, $a \geqslant b > 0$, c, d positive real numbers with $bc \geqslant ad$ and $\psi_q(b+ax) > 0$ then*

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \leqslant \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \leqslant \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d}. \tag{1.7}$$

which again by letting q to 1 gives q-gamma analogue of inequality given in [4] and thus gives a generalization of the main results of [4].

The idea of this paper is to consider the q-gamma analogue of the function given by Theorem 1.1, so to consider the function:

$$f(x) = \frac{\Gamma_q(a+bx)^c}{\Gamma_q(d+ex)^f}, \quad x \geqslant 0 \tag{1.8}$$

and to have q-analogue results of [5] and thus to generalize the results of [9] and [10].

2. Results

In order to establish the proof of the theorems, we need the following lemmas:

Lemma 2.1. *The q-psi function has the following series representation:*

$$\psi_q(x) = -\log(1-q) + \log q \cdot \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}}. \tag{2.1}$$

Proof. See [7]. □

Lemma 2.2. *Let $q \in (0,1)$, $x > 0$, $y > 0$ and $x < y$. Then*

$$\psi_q(x) < \psi_q(y). \tag{2.2}$$

Proof. Using Lemma 2.1 we obtain:

$$\begin{aligned}
\psi_q(x) - \psi_q(y) &= \log q \cdot \Big(\sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}} - \sum_{i=0}^{\infty} \frac{q^{y+i}}{1-q^{y+i}} \Big) \\
&= \log q \cdot \sum_{i=0}^{\infty} \Big(\frac{q^{x+i}}{1-q^{x+i}} - \frac{q^{y+i}}{1-q^{y+i}} \Big) \\
&= \log q \cdot \sum_{i=0}^{\infty} \frac{q^{x+i} - q^{y+i}}{(1-q^{x+i})(1-q^{y+i})} \\
&= \log q \cdot \sum_{i=0}^{\infty} \frac{q^{i}(q^{x} - q^{y})}{(1-q^{x+i})(1-q^{y+i})} < 0,
\end{aligned}$$

because for $x < y$ and $q \in (0,1)$ we have $q^x > q^y$ and $\log q < 0$ which completes the proof. □

Lemma 2.3. *Let $q \in (0,1)$, $a + bx > 0$, $d + ex > 0$ and $a + bx \leqslant d + ex$. Then*

$$\psi_q(a+bx) - \psi_q(d+ex) \leqslant 0. \tag{2.3}$$

Proof. By Lemma 2.2. □

Lemma 2.4. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0$, $d + ex > 0$, $a + bx \leqslant d + ex$ and $ef \geqslant bc > 0$. Let $q \in (0, 1)$. If*

(i) $\psi_q(a + bx) > 0$ *or*

(ii) $\psi_q(d + ex) > 0$

then

$$bc\psi_q(a + bx) - ef\psi_q(d + ex) \leqslant 0. \tag{2.4}$$

Proof. (i) Let $\psi_q(a+bx) > 0$. From Lemma 2.3 we have $\psi_q(d+ex) \geqslant \psi_q(a+bx) > 0$. Multiplying both sides of inequality $ef \geqslant bc$ with $\psi_q(d + ex)$ we obtain

$$ef\psi_q(d + ex) \geqslant bc\psi_q(d + ex) \geqslant bc\psi_q(a + bx),$$

so

$$bc\psi_q(a + bx) - ef\psi_q(d + ex) \leqslant 0.$$

(ii) If $\psi_q(d + ex) > 0$, considering (2.3) we see that there are two possibilities for $\psi_q(a + bx)$.

Case 1. $\psi_q(a + bx) < 0$, *Case 2.* $\psi_q(a + bx) > 0$.

Hence we have:

Case 1. $bc\psi_q(a + bx) < 0$ and $ef\psi_q(d + ex) > 0$ so clearly (2.4) holds.

Case 2. The possibility $\psi_q(a + bx) > 0$ was proved in (i). □

Lemma 2.5. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \leqslant d + ex$ and $bc \geqslant ef > 0$. Let $q \in (0, 1)$. If*

(i) $\psi_q(d + ex) < 0$ *or*

(ii) $\psi_q(a + bx) < 0$

then

$$bc\psi_q(a + bx) - ef\psi_q(d + ex) \leqslant 0. \tag{2.5}$$

Proof. (i) Let $\psi_q(d+ex) < 0$. From Lemma 2.3 we have $\psi_q(a+bx) \leqslant \psi_q(d+ex) < 0$. Multiplying both sides of inequality $bc \geqslant ef$ with $\psi_q(a + bx)$ we obtain

$$bc\psi_q(a + bx) \leqslant ef\psi_q(a + bx) \leqslant ef\psi_q(d + ex),$$

so

$$bc\psi_q(a + bx) - ef\psi_q(d + ex) \leqslant 0.$$

(ii) If $\psi_q(a + bx) < 0$, considering (2.3) we find out that there are two possibilities for $\psi_q(d + ex)$.

Case 1. $\psi_q(d + ex) > 0$, *Case 2.* $\psi_q(d + ex) < 0$.

Then we proceed in the same way as in previous lemma. □

Theorem 2.6. *Let f be a function defined by*

$$f(x) = \frac{\Gamma_q(a+bx)^c}{\Gamma_q(d+ex)^f}, \quad x \geqslant 0,\ q \in (0,1) \tag{2.6}$$

where a,b,c,d,e,f are real numbers such that: $a+bx>0$, $d+ex>0$, $a+bx \leqslant d+ex$, $ef \geqslant bc > 0$. If $\psi_q(a+bx)>0$ or $\psi_q(d+ex)>0$ then the function f is decreasing for $x \geqslant 0$. For $x \in [0,1]$ the following double inequality holds:

$$\frac{\Gamma_q(a+b)^c}{\Gamma_q(d+e)^f} \leqslant \frac{\Gamma_q(a+bx)^c}{\Gamma_q(d+ex)^f} \leqslant \frac{\Gamma_q(a)^c}{\Gamma_q(d)^f}. \tag{2.7}$$

Proof. Let g be a function defined by $g(x) = \log f(x)$. Then

$$g(x) = c \log \Gamma_q(a+bx) - f \log \Gamma_q(d+ex).$$

So

$$g'(x) = bc\frac{\Gamma_q'(a+bx)}{\Gamma_q(a+bx)} - ef\frac{\Gamma_q'(d+ex)}{\Gamma_q(d+ex)} = bc\psi_q(a+bx) - ef\psi_q(d+ex).$$

By (2.4), we have $g'(x) \leqslant 0$. It means g is decreasing for $x \geqslant 0$, hence f is decreasing for $x \geqslant 0$. For $x \in [0,1]$ we have $f(1) \leqslant f(x) \leqslant f(0)$ or

$$\frac{\Gamma_q(a+b)^c}{\Gamma_q(d+e)^f} \leqslant \frac{\Gamma_q(a+bx)^c}{\Gamma_q(d+ex)^f} \leqslant \frac{\Gamma_q(a)^c}{\Gamma_q(d)^f}.$$

This concludes the proof of the Theorem. □

In a similar way, using Lemma 2.5 it is easy to prove the following theorem.

Theorem 2.7. *Let f be a function defined by*

$$f(x) = \frac{\Gamma_q(a+bx)^c}{\Gamma_q(d+ex)^f} \quad x \geqslant 0, q \in (0,1) \tag{2.8}$$

where a,b,c,d,e,f are real numbers such that: $a+bx>0$, $d+ex>0$, $a+bx \leqslant d+ex$, $bc \geqslant ef > 0$. If $\psi_q(d+ex)<0$ or $\psi_q(a+bx)<0$ then the function f is decreasing for $x \geqslant 0$. For $x \in [0,1]$ the inequality (2.7) holds.

By Theorems 2.6 and 2.7 and using (1.3) it is easy to verify that the following remarks hold:

Remark 2.8. Considering (2.7) with $a=1$, $b=1$, $c=n$, $n \in \mathbb{N}$, $d=1$, $e=n$, $n \in \mathbb{N}$, $f=1$ and (1.3) one obtains the q-analogue to the inequality given in [1], which was proved in [9]

Remark 2.9. Considering (2.7) with $a=1$, $b=1$, $c=a$, $a \geqslant 1$, $d=1$, $e=a$, $f=1$ and (1.3) one obtains the q-analogue to the inequality given in [2], also proved in [9].

Remark 2.10. If in (2.7) we take $a = 1$, $c = a$, $d = 1$, $e = a$, $f = b$, with $c \geqslant f > 0$ and using (1.3) we obtain q-analogue to the inequality given in [3].

Remark 2.11. If in (2.7) we take $a = b$, $b = a$, $c = d$, $d = a$, $e = b$, $f = c$, $ef \geqslant bc > 0$, with $a \geqslant b > 0$ and $\psi_q(b + ax) > 0$, as well as using (1.3) we obtain q-analogue to the inequality [4] proved in [10].

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