

Connection between ordinary multinomials, Fibonacci numbers, Bell polynomials and discrete uniform distribution*

Hacène Belbachir, Sadek Bouroubi, Abdelkader Khelladi

Faculty of Mathematics, University of Sciences and Technology Houari Boumediene
(U.S.T.H.B), Algiers, Algeria.

Submitted 8 July 2008; Accepted 16 September 2008

Abstract

Using an explicit computable expression of ordinary multinomials, we establish three remarkable connections, with the q -generalized Fibonacci sequence, the exponential partial Bell partition polynomials and the density of convolution powers of the discrete uniform distribution. Identities and various combinatorial relations are derived.

Keywords: Ordinary multinomials, Exponential partial Bell partition polynomials, Generalized Fibonacci sequence, Convolution powers of discrete uniform distribution.

MSC: 05A10, 11B39, 11B65, 60C05

1. Introduction

Ordinary multinomials are a natural extension of binomial coefficients, for an appropriate introduction of these numbers see Smith and Hogatt [18], Bollinger [6] and Andrews and Baxter [2]. These coefficients are defined as follows: Let $q \geq 1$ and $L \geq 0$ be integers. For an integer $a = 0, 1, \dots, qL$, the ordinary multinomial $\binom{L}{a}_q$ is the coefficient of the a -th term of the following multinomial expansion

$$(1 + x + x^2 + \dots + x^q)^L = \sum_{a \geq 0} \binom{L}{a}_q x^a, \quad (1.1)$$

with $\binom{L}{a}_1 = \binom{L}{a}$ (being the usual binomial coefficient) and $\binom{L}{a}_q = 0$ for $a > qL$.

*Research supported partially by LAID3 Laboratory of USTHB University.

Using the classical binomial coefficient, one has

$$\binom{L}{a}_q = \sum_{j_1+j_2+\dots+j_q=a} \binom{L}{j_1} \binom{L}{j_2} \dots \binom{L}{j_q}. \quad (1.2)$$

Some readily well known established properties are

the symmetry relation

$$\binom{L}{a}_q = \binom{L}{qL-a}_q, \quad (1.3)$$

the longitudinal recurrence relation

$$\binom{L}{a}_q = \sum_{m=0}^q \binom{L-1}{a-m}_q, \quad (1.4)$$

and the diagonal recurrence relation

$$\binom{L}{a}_q = \sum_{m=0}^L \binom{L}{m} \binom{m}{a-m}_{q-1}. \quad (1.5)$$

These coefficients, as for usual binomial coefficients, are built through the Pascal triangle, known as ‘‘Generalized Pascal Triangle’’, see tables: 1, 2 and 3. One can find the first values of the generalized triangle in SLOANE [17] as A027907 for $q = 2$, A008287 for $q = 3$ and A035343 for $q = 4$.

As an illustration of recurrence relation, we give the triangles of trinomial, quadrinomial and pentanomial coefficients:

Table 1: Triangle of *trinomial* coefficients: $\binom{L}{a}_2$

$L \setminus a$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	1	2	3	2	1						
3	1	3	6	7	6	3	1				
4	1	4	10	16	19	16	10	4	1		
5	1	5	15	30	45	51	45	30	15	5	1

Table 2: Triangle of *quadrinomial* coefficients: $\binom{L}{a}_3$

$L \setminus a$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1	1									
2	1	2	3	4	3	2	1						
3	1	3	6	10	12	12	10	6	3	1			
4	1	4	10	20	31	40	44	40	31	20	10	4	1

Table 3: Triangle of *pentanomial* coefficients: $\binom{L}{a}_4$

$L \setminus a$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0	1														
1	1	1	1	1	1										
2	1	2	3	4	5	4	3	2	1						
3	1	3	6	10	15	18	19	18	15	10	6	3	1		
4	1	4	10	20	35	52	68	80	85	80	68	52	35	20	...

Several extensions and commentaries about these numbers have been investigated in the literature, for example Brondarenko [7] gives a combinatorial interpretation of ordinary multinomials $\binom{L}{a}_q$ as *the number of different ways of distributing “a” balls among “L” cells where each cell contains at most “q” balls.*

Using this combinatorial argument, one can easily establish the following relation

$$\begin{aligned} \binom{L}{a}_q &= \sum_{L_1+2L_2+\dots+qL_q=a} \binom{L}{L_1} \binom{L-L_1}{L_2} \dots \binom{L-L_1-\dots-L_{q-1}}{L_q} \\ &= \sum_{L_1+2L_2+\dots+qL_q=a} \binom{L}{L_1, L_2, \dots, L_q}. \end{aligned} \tag{1.6}$$

For a computational view of the relation (1.6) see Bollinger [6]. Andrews and Baxter [2] have considered the q -analog generalization of ordinary multinomials (see also [19] for an exhaustive bibliography). They have defined the q -multinomial coefficients as follows

$$\binom{L}{a}_q^{(p)} = \sum_{j_1+j_2+\dots+j_q=a} q^{\sum_{i=1}^{q-1} (L-j_i)j_{i+1} - \sum_{i=q-p}^{q-1} j_{i+1}} \begin{bmatrix} L \\ j_1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \dots \begin{bmatrix} j_{q-1} \\ j_q \end{bmatrix}$$

where

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{bmatrix} L \\ a \end{bmatrix}_q = \begin{cases} (q)_L / (q)_a (q)_{L-a} & \text{if } 0 \leq a \leq L \\ 0 & \text{otherwise} \end{cases}$$

is the usual q -binomial coefficient, and where $(q)_k = \prod_{m=1}^{\infty} (1 - q^m) / (1 - q^{k+m})$, is called q -series. This definition is motivated by the relation (1.2).

Another extension, the supernomials, has also been considered by Schilling and Warnaar [16]. These coefficients are defined to be the coefficients of x^a in the expression of $\prod_{j=1}^N (1 + x + \dots + x^j)^{L_j}$

A refinement of the q -multinomial coefficient is also considered for the trinomial case by Warnaar [20].

Barry [3] gives a generalized Pascal triangle as

$$\binom{n}{k}_{a(n)} := \prod_{j=1}^k a(n-j+1) / a(j),$$

where $a(n)$ is a suitably chosen sequence of integers.

Kallas [11] and Noe [14] give a generalization of Pascal's triangle by considering the coefficient of x^a in the expression of $(a_0 + a_1x + \dots + a_qx^q)^L$.

The main goal of this paper is to give some connections of the ordinary multinomials with the generalized Fibonacci sequence, the exponential Bell polynomials, and the density of convolution powers of discrete uniform distribution. We will give also some interesting combinatorial identities.

2. A simple expression of ordinary multinomials

If we denote x_i the number of balls in a cell, the previous combinatorial interpretation given by Brondarenko is equivalent to evaluate the number of solutions of the system

$$\begin{cases} x_1 + \dots + x_L = a, \\ 0 \leq x_1, \dots, x_L \leq q. \end{cases} \quad (2.1)$$

Now, let us consider the system (2.1). For $t \in]-1, 1[$, we have (see also Comtet [8, Vol. 1, p. 92 (pb 16).])

$$\sum_{a \geq 0} \binom{L}{a}_q t^a = (1 + t + \dots + t^q)^L = \sum_{0 \leq x_1, \dots, x_L \leq q} t^{x_1 + \dots + x_L},$$

and

$$\begin{aligned} (1 + t + \dots + t^q)^L &= (1 - t^{q+1})^L (1 - t)^{-L} \\ &= \left(\sum_{j=0}^L (-1)^j \binom{L}{j} t^{j(q+1)} \right) \left(\sum_{j \geq 0} \binom{j+L-1}{L-1} t^j \right). \end{aligned}$$

By identification, we obtain the following theorem.

Theorem 2.1. *The following identity holds*

$$\binom{L}{a}_q = \sum_{j=0}^{\lfloor a/(q+1) \rfloor} (-1)^j \binom{L}{j} \binom{a - j(q+1) + L - 1}{L - 1}. \quad (2.2)$$

This explicit relation seems to be important since in contrast to relations (1.2), (1.3) and (1.5), it allows to compute the ordinary multinomials with one summation symbol.

In 1711, de Moivre (see [13] or [12, 3rd ed. p. 39]) solves the system (2.1) as the right hand side of (2.2).

Corollary 2.2. *We have the following identity*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \binom{n-j}{j} = \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^j \binom{n}{j} \binom{2n - 3j - 1}{n - 1}.$$

Proof. It suffices to use relation (6) in Theorem 2.1 for $q = 2$ and $a = L = n$. \square

The left hand side of the equality has the following combinatorial meaning. It computes the number of ways to distribute n balls into n boxes with 2 balls at most into each box. Put a ball into each box, then choose j boxes for removing the boxes located in them into j boxes chosen from the remaining $n - j$ boxes.

3. Generalized Fibonacci sequences

Now, let us consider for $q \geq 1$, the “multibonacci” sequence $(\Phi_n^{(q)})_{n \geq -q}$ defined by

$$\begin{cases} \Phi_{-q}^{(q)} = \dots = \Phi_{-2}^{(q)} = \Phi_{-1}^{(q)} = 0, \\ \Phi_0^{(q)} = 1, \\ \Phi_n^{(q)} = \Phi_{n-1}^{(q)} + \Phi_{n-2}^{(q)} + \dots + \Phi_{n-q-1}^{(q)} \text{ for } n \geq 1. \end{cases}$$

In [4], Belbachir and Bencherif proved that

$$\Phi_n^{(q-1)} = \sum_{k_1+2k_2+\dots+qk_q=n} \binom{k_1+k_2+\dots+k_q}{k_1, k_2, \dots, k_q},$$

and, for $n \geq 1$

$$\Phi_n^{(q-1)} = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} (-1)^k \frac{n-k(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)},$$

leading to

$$\sum_{k_1+\dots+qk_q=n} \binom{k_1+\dots+k_q}{k_1, \dots, k_q} = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} (-1)^k \frac{n-k(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)}.$$

This is an analogous situation in writing above a multiple summation with one symbol of summation. On the other hand, we establish a connection between the ordinary multinomials and the generalized Fibonacci sequence:

Theorem 3.1. *We have the following identity*

$$\Phi_n^{(q)} = \sum_{l=0}^{qm-r} \binom{n-l}{l}_q, \tag{3.1}$$

where m is given by the extended euclidean algorithm for division: $n = m(q+1) - r$, $0 \leq r \leq q$.

Proof. We have

$$\begin{aligned}
 \Phi_n^{(q)} &= \sum_{k_1+2k_2+\dots+(q+1)k_{q+1}=n} \binom{k_1+k_2+\dots+k_{q+1}}{k_1, k_2, \dots, k_{q+1}} \\
 &= \sum_{L \geq 0} \sum_{k_1+2k_2+\dots+(q+1)k_{q+1}=n} \binom{L}{k_1, k_2, \dots, k_{q+1}} \\
 &= \sum_{L \geq 0} \sum_{k_2+2k_3+\dots+qk_{q+1}=n-L} \binom{L}{L-k_2-\dots-k_{q+1}, k_2, \dots, k_{q+1}} \\
 &= \sum_{L \geq 0} \binom{L}{n-L}_q \\
 &= \sum_{L \geq \frac{n}{q+1}}^n \binom{L}{n-L}_q,
 \end{aligned}$$

using the fact that $\binom{L}{a}_q = 0$ for $a < 0$ or $a > qL$

Now consider the unique writing of n given by the extended euclidean algorithm for division: $n = m(q+1) - r$, $0 \leq r < q+1$ then $\frac{n}{q+1} = m - \frac{r}{q+1}$, which gives

$$\Phi_n^{(q)} = \sum_{k=0}^{qm-r} \binom{m+k}{qm-r-k}_q = \sum_{k=0}^{qm-r} \binom{m+k}{(q+1)k+r}_q = \sum_{l=0}^{qm-r} \binom{n-l}{l}_q.$$

□

As an immediate consequence of Theorem 3.1, we obtain the following identities

$$\begin{aligned}
 \Phi_{(q+1)m}^{(q)} &= \sum_{l=0}^{qm} \binom{(q+1)m-l}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)k}_q, \\
 \Phi_{(q+1)m-1}^{(q)} &= \sum_{l=0}^{qm-1} \binom{(q+1)m-l-1}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)k+1}_q, \\
 &\vdots \\
 \Phi_{(q+1)m-r}^{(q)} &= \sum_{l=0}^{qm-r} \binom{(q+1)m-l-r}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)k+r}_q.
 \end{aligned}$$

For $q = 1$, we find the classical Fibonacci sequence:

$$F_{-1} = 0, F_0 = 1, F_{n+1} = F_n + F_{n-1}, \text{ for } n \geq 0.$$

Thus, we obtain the well known identity

$$F_n = \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l}{l}.$$

Recently, in [5], the first author and Szalay prove the unimodality of the sequence $u_k = \binom{n-k}{k}_q$ associated to generalized Fibonacci numbers. More generally, they establish the unimodality for all rays of generalized Pascal triangles by showing that the sequence $w_k = \binom{n+\alpha k}{m+\beta k}_q$ is log-concave, then unimodal.

4. Exponential partial Bell partition polynomials

In this section, we establish a connection of the ordinary multinomials with exponential partial Bell partition polynomials $B_{n,L}(t_1, t_2, \dots)$ which are defined (see Comtet [8, p. 144]) as follows

$$\frac{1}{L!} \left(\sum_{m \geq 1} \frac{t_m}{m!} x^m \right)^L = \sum_{n \geq L} B_{n,L} \frac{x^n}{n!}, \quad L = 0, 1, 2, \dots \tag{4.1}$$

An exact expression of such polynomials is given by

$$B_{n,L}(t_1, t_2, \dots) = \sum_{\substack{k_1+2k_2+\dots=n \\ k_1+k_2+\dots=L}} \frac{n!}{k_1!k_2! \dots (1!)^{k_1} (2!)^{k_2} \dots} t_1^{k_1} t_2^{k_2} \dots$$

In this expression, the number of variables is finite according to $k_1 + 2k_2 + \dots = n$.

Next, we give some particular values of $B_{n,L}$:

$$\begin{aligned} B_{n,L}(1, 1, 1, \dots) &= \left\{ \begin{matrix} n \\ L \end{matrix} \right\} \text{ Stirling numbers of second kind,} \\ B_{n,L}(0!, 1!, 2!, \dots) &= \left[\begin{matrix} n \\ L \end{matrix} \right] \text{ Stirling numbers of first kind,} \\ B_{n,L}(1!, 2!, 3!, \dots) &= \frac{n!}{L!} \binom{n-1}{n-L}. \end{aligned} \tag{4.2}$$

In [1], Abbas and Bouroubi give several extended values of $B_{n,L}$.

The connection with ordinary multinomials is given by the following result:

Theorem 4.1. *We have the following identity*

$$B_{n,L}(1!, 2!, \dots, (q+1)!, 0, \dots) = \frac{n!}{L!} \binom{L}{n-L}_q. \tag{4.3}$$

Proof. Taking in (4.1) $t_m = m!$ for $1 \leq m \leq q+1$ and zero otherwise, we obtain

$$(x + \dots + x^{q+1})^L = L! \sum_{n-L \geq 0} B_{n,L}(1!, 2!, \dots, (q+1)!, 0, \dots) \frac{x^n}{n!},$$

from which it follows

$$\sum_{a \geq 0} \binom{L}{a}_q x^a = \sum_{n-L \geq 0} \frac{L!}{n!} B_{n,L}(1!, 2!, \dots, (q+1)!, 0, \dots) x^{n-L}.$$

□

Corollary 4.2. *Let $q \geq 1$, $L \geq 0$ be integers, and $a \in \{0, 1, \dots, qL\}$. For $q \geq a$, we have the following identity*

$$\binom{L}{a}_q = \binom{L+a-1}{a}.$$

Proof. Using the fact that $B_{n,L}(1!, 2!, \dots, (q+1)!, 0, \dots) = B_{n,L}(1!, 2!, 3!, \dots)$ for $q+1 \geq n-L+1$, we obtain $\binom{L}{n-L}_q = \binom{n-1}{n-L}$ for $q \geq n-L$. We conclude with $a = n-L$. □

This is simply a combination with repetition permitted (i.e. multi combination).

5. Convolution powers of discrete uniform distribution

This section gives a connection between the ordinary multinomials and the convolution power of the discrete uniform distribution. The right hand side of identity (2.2) is a very well known expression. Indeed for $q, L \in \mathbb{N}$, let us denote by U_q^{*L} the L^{th} convolution power of the discrete uniform distribution

$$U_q := \frac{1}{q+1} (\delta_0 + \delta_1 + \dots + \delta_q) \quad (\delta_a \text{ is the Dirac measure}),$$

then for $a \in \mathbb{N}$ (see de Moivre [13] or [10]), with respect to the counting measure, its density is given by

$$P(U_q^{*L} = a) = \frac{1}{(q+1)^L} \sum_{j=0}^{\lfloor a/(q+1) \rfloor} (-1)^j \binom{L}{j} \binom{a+L-(q+1)j-1}{L-1}. \quad (5.1)$$

Combining Theorem 2.1 and relation (5.1), we have the following result:

Corollary 5.1. *Using the above notations, we obtain the following identity*

$$P(U_q^{*L} = a) = \frac{\binom{L}{a}_q}{(q+1)^L}.$$

It should be noted that the multinomials may be seen as the number of favorable cases to the realization of the elementary event $\{a\}$.

It is easy to show that the distribution of U_q^{*L} is symmetric by relation (1.3).

Corollary 5.2. *We have the following identities*

$$\begin{aligned} \sum_{k=0}^{qL} k \binom{L}{k}_q &= (q+1)^L \frac{qL}{2}, \\ \sum_{k=0}^{qL} k^2 \binom{L}{k}_q &= (q+1)^L \frac{qL}{2} \left(\frac{qL}{2} + \frac{q+2}{6} \right), \\ \sum_{k=0}^{qL} k^3 \binom{L}{k}_q &= (q+1)^L \left(\frac{qL}{2} \right)^2 \left(\frac{qL}{2} + \frac{q+2}{2} \right), \end{aligned}$$

More generally, for $m \geq 1$, the following identity holds

$$\sum_{k=0}^{qL} k^m \binom{L}{k}_q = (q+1)^L \sum_{i_1+i_2+\dots+i_L=m} \binom{m}{i_1, i_2, \dots, i_L} u_{i_1} u_{i_2} \dots u_{i_L},$$

where u_i is the i -th moment of the random variable U_q .

Proof. It suffices to compute the expectation of U_q^{*L} using, first the density distribution and second the summation of uniform distributions. It also comes from the application of the generating function of the distribution given by Corollary 5.1. \square

Acknowledgements. The authors are grateful to Professor Miloud Mihoubi for pointing our attention to Bell polynomials. The authors are also grateful to the referee and would like to thank him/her for comments and suggestions which improved the quality of this paper.

References

- [1] ABBAS, M., BOUROUBI, S., On new identities for Bell's polynomials, *Disc. Math.*, 293 (2005) 5–10.
- [2] ANDREWS, G.E., BAXTER, J., Lattice gas generalization of the hard hexagon model III q -trinomial coefficients, *J. Stat. Phys.*, 47 (1987) 297–330.
- [3] BARRY, P., On Integer-sequences-based constructions of generalized Pascal triangles. *Journal of integer sequences*, Vol. 9 (2006), Art. 06.2.4.
- [4] BELBACHIR, H., BENCHERIF, F., Linear recurrent sequences and powers of a square matrix. *Integers*, 6 (2006), A12, 17 pp.
- [5] BELBACHIR, H., SZALAY, L., Unimodal rays in the regular and generalized Pascal triangles, *J. of Integer Seq.*, Vol. 11, Art. 08.2.4. (2008).

- [6] BOLLINGER, R.C., A note on Pascal T -triangles, Multinomial coefficients and Pascal Pyramids, *The Fibonacci Quarterly*, 24 (1986) 140–144.
- [7] BRONDARENKO, B.A., Generalized Pascal triangles and Pyramids, their fractals, graphs and applications, *The Fibonacci Association*, Santa Clara 1993, Translated from Russian by R.C. Bollinger.
- [8] COMTET, L., Analyse combinatoire, *Puf, Coll. Sup. Paris*, (1970), Vol. 1 & Vol. 2.
- [9] GRAHAM, R.L., KNUTH, D.E., PATASHNIK, O., Concrete mathematics, *Addison-Wesley*, 1994.
- [10] HALD, A., A history of mathematical statistics from 1750 to 1930, *John Wiley, N. Y.*, 1998.
- [11] KALLOS, G., A generalization of Pascal triangles using powers of base numbers; *Annales mathématiques Blaise Pascal*, Vol. 13, no. 1 (2006) 1–15.
- [12] DE MOIVRE, A., The doctrine of chances, Third edition 1756 (first ed. 1718 and second ed. 1738), reprinted by *Chelsea, N. Y.*, 1967.
- [13] DE MOIVRE, A., Miscellanea Analytica de Scricus et Quadraturis, *J. Tomson and J. Watts, London*, 1731.
- [14] NOE, T.D., On the divisibility of generalized central trinomial coefficients, *Journal of Integer sequences*, Vol. 9 (2006) Art. 06.2.7.
- [15] PHILIPPOU, A.N., A note of the Fibonacci sequence of order k and the Multinomial coefficients, *The Fibonacci Quarterly*, 21 (1983) 82–86.
- [16] SCHILLING, A., WARNAAR, S.O., Supernomial coefficients, Polynomial identities and q -series, *The Ramanujan J.*, 2 (1998) 459–494.
- [17] SLOANE, N.J.A., The online Encyclopedia of Integer sequences, Published electronically at <http://www.research.att.com/~njas/sequences>, 2008.
- [18] SMITH, C., HOGATT, V.E., Generating functions of central values of generalized Pascal triangles, *The Fibonacci Quarterly*, 17 (1979) 58–67.
- [19] WARNAAR, S.O., The Andrews-Gordon Identities and q -Multinomial coefficients, *Commun. Math. Phys.*, 184 (1997) 203–232.
- [20] WARNAAR, S.O., Refined q -trinomial coefficients and character identities, *J. Statist. Phys.*, 102 (2001), no. 3–4, 1065–1081.

Hacène Belbachir

Sadek Bouroubi

Abdelkader Khelladi

USTHB, Faculté de Mathématiques

BP 32, El Alia

16111 Bab Ezzouar

Alger, Algérie

e-mail:

hbelbachir@usthb.dz, hacenebelbachir@gmail.com

sbouroubi@usthb.dz, bouroubis@yahoo.fr

akhelladi@usthb.dz, khelladi@wissal.dz