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A general method to obtain the rate of convergence in the strong law of large numbers

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Abstract

A general approach to the rate of convergence in the strong law of large numbers is given. It is based on the Hájek–Rényi type method presented in Sung, Hu and Volodin [5].

Keywords: strong law of large numbers, Hájek–Rényi maximal inequality, rate of convergence

MSC: 60F15

1. Introduction

The Hájek–Rényi inequality (see Hájek and Rényi [3]) is a useful tool to prove the strong law of large numbers (SLLN). There are several generalizations of that inequality. In Fazekas and Klesov [1] a unified approach is given to obtain SLLN's. Their method is based on a Hájek–Rényi type inequality for the moments. Then the general method is applied to prove SLLN's for various dependent sequences. It turned out that by their method the normalizing constants in the SLLN's can be improved. Hu and Hu in [4] strengthened the method of Fazekas and Klesov [1] by adding the rate of convergence in the SLLN.

Sung, Hu and Volodin [5] found a new method for obtaining the strong growth rate for sums of random variables by using the method of Fazekas and Klesov [1]. This result generalizes and sharpens the method of Hu and Hu [4].

Tómács and Líbor in [6] gave a version of the approach in Fazekas and Klesov [1] by using Hájek–Rényi type inequality for the probabilities instead of the moments.

In this paper we give a general method to obtain the rate of convergence in an SLLN by using a Hájek–Rényi type inequality for the probabilities (see Theorem 3.4). This result generalizes the method of Sung, Hu and Volodin [5].

We use the following notation. Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. If $a_1, a_2, \ldots \in \mathbb{R}$ then in case $A = \emptyset$ let $\max_{k\in A} a_k = 0$ and $\sum_{k\in A} a_k = 0$. Let Ψ denote the set of functions $f\colon (0,\infty) \to (0,\infty)$, that are nonincreasing and

$$\sum_{n=1}^{\infty} n^{-2} f(n^{-1}) < \infty.$$

2. Lemmas

Lemma 2.1. *If $f \in \Psi$ then $\sum_{k=0}^{\infty} 2^{-k} f(2^{-k}) < \infty$.*

Proof. It is easy to see that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-2} f(n^{-1}) &= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{-2} f(n^{-1})\\ &\geqslant \sum_{k=0}^{\infty} f(2^{-k}) \sum_{n=2^k}^{2^{k+1}-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k} f(2^{-k}).\end{aligned}$$

This inequality implies the statement. □

The following lemma generalizes Dini's theorem (see Fikhtengolts [2], §375.5 or Lemma 1 in Hu and Hu [4]).

Lemma 2.2. *Let $\{a_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative numbers such that $a_k > 0$ for infinitely many k. Let $f \in \Psi$. If $\sum_{k=1}^{\infty} a_k < \infty$ then*

$$\sum_{k=1}^{\infty} a_k f\left(\sum_{i=k}^{\infty} a_i\right) < \infty.$$

Proof. Let $v_k = \sum_{i=k}^{\infty} a_i$. Then $\{v_k,\ k \in \mathbb{N}\}$ is a nonincreasing sequence of positive numbers and $\lim_{k\to\infty} v_k = 0$.

Let $A_i = \{k \in \mathbb{N} \ :\ 2^{-i-1} < v_k \leqslant 2^{-i}\}$, $i = 0, 1, 2, \ldots$, and $k_0 = \min \bigcup_{i=0}^{\infty} A_i$. If $A_i \neq \emptyset$, then with notation $m_i = \min A_i$, we have

$$\sum_{k\in A_i} a_k \leqslant \sum_{k=m_i}^{\infty} a_k = v_{m_i} \leqslant 2^{-i}.$$

So we get

$$\sum_{k=k_0}^{\infty} a_k f(v_k) = \sum_{i=0}^{\infty} \sum_{k\in A_i} a_k f(v_k) \leqslant \sum_{i=0}^{\infty} f(2^{-i-1}) \sum_{k\in A_i} a_k$$

$$\leqslant \sum_{i=0}^{\infty} 2^{-i} f(2^{-i-1}) = 2 \sum_{i=1}^{\infty} 2^{-i} f(2^{-i})$$

which is less then ∞ by Lemma 2.1. Thus the statement is proved. $\square$

Lemma 2.3. *Let* $\{Y_k,\ k \in \mathbb{N}\}$ *be a sequence of random variables defined on a fixed probability space* $(\Omega, \mathcal{F}, \mathrm{P})$. *Then*

$$\mathrm{P}\Big(\sup_k Y_k > x\Big) = \lim_{n\to\infty} \mathrm{P}\Big(\max_{k\leqslant n} Y_k > x\Big) \quad \textit{for all} \quad x \in \mathbb{R}.$$

Proof. It is easy to see that

$$\Big\{\sup_k Y_k > x\Big\} = \bigcup_{n=1}^{\infty} \Big\{\max_{k\leqslant n} Y_k > x\Big\} \quad \text{for all} \quad x \in \mathbb{R},$$

hence, using continuity of probability, we get the statement. $\square$

Lemma 2.4. *Let* $\{Y_k,\ k \in \mathbb{N}\}$ *be a sequence of random variables defined on a fixed probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *and* $\{\varepsilon_n,\ n \in \mathbb{N}\}$ *a nondecreasing sequences of real numbers. If*

$$\lim_{n\to\infty} \mathrm{P}\Big(\sup_k Y_k > \varepsilon_n\Big) = 0,$$

then

$$\sup_k Y_k < \infty \quad \textit{almost surely} \ (\textit{a.s.}).$$

Proof. Using continuity of probability, we have

$$\mathrm{P}\left(\bigcap_{n=1}^{\infty} \{\sup_k Y_k > \varepsilon_n\}\right) = \lim_{n\to\infty} \mathrm{P}\Big(\sup_k Y_k > \varepsilon_n\Big) = 0,$$

which is equivalent to $\mathrm{P}\Big(\bigcup_{n=1}^{\infty}\{\sup_k Y_k \leqslant \varepsilon_n\}\Big) = 1$. This implies that there exists $n_\omega \in \mathbb{N}$ for almost every $\omega \in \Omega$, such that $\sup_k Y_k(\omega) \leqslant \varepsilon_{n_\omega} < \infty$. $\square$

3. The general method

In this section let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and $S_n = \sum_{k=1}^{n} X_k$ for all $n \in \mathbb{N}$. Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers, $r > 0$ and $\{b_k,\ k \in \mathbb{N}\}$ a nondecreasing unbounded sequence of positive real numbers. Assume that

$$\sum_{k=1}^{\infty} \alpha_k b_k^{-r} < \infty$$

and there exists $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k\leqslant n}|S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r}\sum_{k=1}^{n}\alpha_k. \tag{3.1}$$

Let $f \in \Psi$, $g(x) = f^{-1/r}(x)$ if $x > 0$, $g(0) = 0$ and

$$\beta_n = \max_{k\leqslant n} b_k g\left(\sum_{i=k}^{\infty}\alpha_i b_i^{-r}\right).$$

Remark 3.1. It is proved that these conditions imply $\lim_{n\to\infty} S_n b_n^{-1} = 0$ a.s. (See Theorem 2.4 in [6].)

Theorem 3.2. *If there exists $t \in \mathbb{N}$ such that $\alpha_t \neq 0$, then*

$$\frac{S_n}{\beta_n} = \begin{cases} O(1) \ \ a.s., & \text{if } \beta_n = O(1),\\ o(1) \ \ a.s., & \text{if } \beta_n \neq O(1).\end{cases}$$

Proof. It is easy to see that $0 < \beta_1 \leqslant \beta_2 \leqslant \cdots$. First we shall prove that

$$\sum_{k=1}^{\infty}\alpha_k\beta_k^{-r} < \infty. \tag{3.2}$$

If $\alpha_k > 0$ for finitely many k, then (3.2) is obvious. If $\alpha_k > 0$ for infinitely many k, then

$$\begin{aligned}\beta_n^{-r} &= \left(\max_{k\leqslant n} b_k f^{-1/r}\Big(\sum_{i=k}^{\infty}\alpha_i b_i^{-r}\Big)\right)^{-r}\\ &\leqslant \left(b_n f^{-1/r}\Big(\sum_{i=n}^{\infty}\alpha_i b_i^{-r}\Big)\right)^{-r} = b_n^{-r} f\left(\sum_{i=n}^{\infty}\alpha_i b_i^{-r}\right).\end{aligned}$$

This inequality and Lemma 2.2 imply

$$\sum_{k=1}^{\infty}\alpha_k\beta_k^{-r} \leqslant \sum_{k=1}^{\infty}\alpha_k b_k^{-r} f\left(\sum_{i=k}^{\infty}\alpha_i b_i^{-r}\right) < \infty.$$

Thus (3.2) is proved. Now, if $\beta_n \neq O(1)$, then Remark 3.1 and (3.2) imply the statement. If $\beta_n = O(1)$, then we get by (3.2)

$$\sum_{k=1}^{\infty}\alpha_k \leqslant \sum_{k=1}^{\infty}\alpha_k\big(\beta_k^{-1}\sup_n\beta_n\big)^r = \big(\sup_n\beta_n\big)^r\sum_{k=1}^{\infty}\alpha_k\beta_k^{-r} < \infty. \tag{3.3}$$

By Lemma 2.3 and (3.1) we have

$$\mathrm{P}\Big(\sup_k|S_k| > \varepsilon\Big) \leqslant \lim_{n\to\infty}\mathrm{P}\Big(\max_{k\leqslant n}|S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r}\sum_{k=1}^{\infty}\alpha_k \quad \text{for all} \quad \varepsilon > 0.$$

This inequality and (3.3) imply

$$\lim_{n\to\infty} \mathrm{P}\Big(\sup_k |S_k| > \varepsilon_n\Big) = 0,$$

where $0 < \varepsilon_n \uparrow \infty$. Hence by Lemma 2.4 we get $\sup_k |S_k| < \infty$ a.s. On the other hand $S_n\beta_n^{-1} \leqslant \sup_k |S_k|\beta_1^{-1}$. Thus the theorem is proved. □

Remark 3.3. Sung, Hu and Volodin proved in [5] (Lemma 4) that if $\alpha_n \equiv 1$ then $\beta_n \neq O(1)$. Hence Theorem 3.2 implies that if $\alpha_n \equiv 1$ then $\lim_{n\to\infty} S_n\beta_n^{-1} = 0$ a.s.

Theorem 3.4. *The following statements are true:*

(1) $S_n b_n^{-1} = O(\beta_n b_n^{-1})$ *a.s.*
(2) $\lim_{n\to\infty} S_n b_n^{-1} = 0$ *a.s.*
(3) *If* $\alpha_k > 0$ *for finitely many* k*, then* $\lim_{n\to\infty} \beta_n b_n^{-1} = 0$.
(4) *If* $\lim_{x\to 0} f(x) = \infty$*, then* $\lim_{n\to\infty} \beta_n b_n^{-1} = 0$.

Proof. Let $w_k = \sum_{i=k}^{\infty} \alpha_i b_i^{-r}$. Then $w_1 \geqslant w_2 \geqslant \dots$ and $\lim_{n\to\infty} w_n = 0$, hence we get

$$\beta_n \leqslant \max_{k<m} b_k g(w_k) + \max_{m\leqslant k\leqslant n} b_k g(w_k) \leqslant \max_{k<m} b_k g(w_k) + b_n g(w_m), \quad \text{if} \quad n > m. \tag{3.4}$$

On the other hand

$$\lim_{n\to\infty} \Big(b_n^{-1} \max_{k<m} b_k g(w_k) + g(w_m)\Big) = g(w_m), \quad \forall m \in \mathbb{N}. \tag{3.5}$$

Now, we shall prove that

$$\lim_{m\to\infty} g(w_m) = 0. \tag{3.6}$$

If $\alpha_k > 0$ for finitely many k, then (3.6) is obvious. If $\alpha_k > 0$ for infinitely many k, then the condition is $\lim_{x\to 0} f(x) = \infty$, which implies $\lim_{m\to\infty} f(w_m) = \infty$. Hence, (3.6) is true in this case too. Then (3.4), (3.5) and (3.6) imply $\lim_{n\to\infty} \beta_n b_n^{-1} = 0$.

Now, we turn to statement $S_n b_n^{-1} = O(\beta_n b_n^{-1})$ a.s. If there exists $t \in \mathbb{N}$ such that $\alpha_t \neq 0$, then by Theorem 3.2 we have

$$\frac{S_n}{b_n} = \frac{S_n}{\beta_n}\frac{\beta_n}{b_n} = O(1)\frac{\beta_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad \text{a.s.}$$

If $\alpha_k \equiv 0$, then by (3.1) we get

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon_m\Big) = 0 \quad \forall m, n \in \mathbb{N},$$

where $0 < \varepsilon_m \downarrow 0$. It follows that $S_n = 0$ a.s. for all $n \in \mathbb{N}$ in this case.

Finally $\lim_{n\to\infty} S_n b_n^{-1} = 0$ a.s. is proved by Tómács and Líbor in [6] (Theorem 2.4). □

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