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On the generalization of the Fibonacci-coefficient polynomials*

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Abstract

In this note we deal with the zeros of polynomials defined recursively, where the coefficients of these polynomials are the terms of a given second order linear recursive sequence of integers. Some results on the Fibonacci-coefficient polynomials obtained by D. Garth, D. Mills and P. Mitchell will be generalized.

Keywords: Fibonacci numbers, polynomials defined recursively, bounds for zeros

MSC: 11C08, 13B25

1. Introduction

Let $R_0 = 0$, $R_1 = 1$, A and B be fixed positive integers and let R_n denote the nth term of the second order linear recursive sequence

$$R = \{R_n\}_{n=0}^{\infty},$$

where for $n \geqslant 2$

$$R_n = AR_{n-1} + BR_{n-2}. \tag{1.1}$$

According to the known Binet-form, for $n \geqslant 0$

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the zeros of the characteristic polynomial $x^2 - Ax - B$ of the sequence R. We can suppose that $\alpha > 0$ and $\beta < 0$.

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In the special case $A = B = 1$ we can get the Fibonacci-sequence, that is, with the usual notation

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 2).$$

We can similarly define the most known second order linear recursive sequences of polynomials, such as the Chebishev-polynomials

$$\{U_n(x)\}_{n=0}^{\infty}$$

of the second kind and the Fibonacci-polynomials

$$\{F_n(x)\}_{n=0}^{\infty},$$

where

$$U_0(x) = 0,\ U_1(x) = 1,\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad (n \geqslant 2)$$

and

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad (n \geqslant 2). \tag{1.2}$$

It is well-known that for $n \geqslant 2$, $U_n(z) = 0$ if and only if $z = \cos\frac{k\pi}{n}$ for $k = 1, 2, \dots, n-1$ and so $z \in \mathbf{R}$ and $|z| < 1$, while for $n \geqslant 2$ $F_n(z') = 0$ if and only if $z' = 2i\cos\frac{k\pi}{n}$ for $k = 1, 2, \dots, n-1$ and so z's are purely imaginary complex numbers except 0 if n is even, and $|z'| < 2$.

According to D. Garth, D. Mills and P. Mitchell [1] the definition of the Fibonacci-coefficient polynomial $p_n(x)$ is the following:

$$p_n(x) = \sum_{k=0}^{n} F_{k+1}x^{n-k} = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}. \tag{1.3}$$

It is worth mentioning that (1.3) is not a suitable (linear) transformation of (1.2).

The aim of this paper is to investigate the zeros of the polynomials $q_n(x)$, where

$$q_n(x) = \sum_{k=0}^{n} R_{k+1}x^{n-k} = R_1x^n + R_2x^{n-1} + \cdots + R_nx + R_{n+1}, \tag{1.4}$$

that is, our results concern to a family of the linear recursive sequences of second order instead of the only one Fibonacci-sequence.

Naturally, with the notation

$$q_n^{\star}(x) = x^n q_n(1/x) = R_1 + R_2x + R_3x^2 + \cdots + R_{n+1}x^n \tag{1.5}$$

we can find information on the zeros of the polynomials $q_n^{\star}(x)$.

2. Preliminary and known results

At first we mention that the polynomials $q_n(x)$ can easily be rewritten in a recursive manner. That is, if $q_0(x) = 1$ then for $n \geqslant 1$

$$q_n(x) = xq_{n-1}(x) + R_{n+1}.$$

Lemma 2.1. *Let for* $n \geqslant 1$, $g_n(x) = (x^2 - Ax - B)q_n(x)$. *Then*

$$g_n(x) = x^{n+2} - R_{n+2}x - BR_{n+1}. \tag{2.1}$$

Proof. Using (1.4) we get $q_1(x) = R_1x + R_2$ and by (1.1) $g_1(x) = (x^2 - Ax - B)q_1(x) = (x^2 - Ax - B)(R_1x + R_2) = \cdots = x^3 - R_3x - BR_2$. Continuing the proof with induction on n, we suppose that the statement is true for $n - 1$ and we prove it for n. Applying (1.4) and (1.1), after some numerical calculations one can get that

$$g_n(x) = (x^2 - Ax - B)q_n(x) = (x^2 - Ax - B)(R_1x^n + R_2x^{n-1} + \cdots + R_nx + R_{n+1})$$

$$= \cdots = x^{n+2} - R_{n+2}x - BR_{n+1}.$$

□

Lemma 2.2 (Theorem of S. Kakeya [3]). *If every coefficients of the polynomial* $f(x) = a_0 + a_1x + \cdots + a_nx^n$ *are positive numbers and the roots of equation* $f(x) = 0$ *are denoted by* $z_1, z_2, \ldots, z_n$, *then*

$$\gamma \leqslant |z_i| \leqslant \delta$$

holds for every $1 \leqslant i \leqslant n$, *where* γ *is the minimal, while* δ *is the maximal value in the sequence*

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}.$$

The following lemma can be found in [2].

Lemma 2.3. *Let us consider the sequence* R *defined by* (1.1). *The increasing order of the elements of the set*

$$\left\{ \frac{R_{i+1}}{R_i} \ : \ 1 \leqslant i \leqslant n \right\}$$

is

$$\frac{R_2}{R_1}, \frac{R_4}{R_3}, \frac{R_6}{R_5}, \ldots, \frac{R_7}{R_6}, \frac{R_5}{R_4}, \frac{R_3}{R_2}.$$

3. Results and proofs

At first we deal with the number of the real zeros of the polynomials $q_n(x)$ defined in (1.4).

Theorem 3.1. a) *If $n \geqslant 2$ and even, then the polynomial $q_n(x)$ has not any real zero, that is, every zeros are non-real complex numbers.*

b) *If $n \geqslant 3$ and odd, then the polynomial $q_n(x)$ has only one real zero and this is negative. That is, every but one zeros are non-real complex numbers.*

Proof. Because of the definition (1.1) of the sequence R the coefficients of the polynomials $q_n(x)$ are positive ones, thus positive real root of the equation $q_n(x) = 0$ does not exist. That is, it is enough to deal with only the existence of negative roots of the equation $q_n(x) = 0$.

a) Since n is even, by (2.1), the coefficients of the polynomial $g_n(-x) = (-x)^{n+2} - R_{n+2}(-x) - BR_{n+1} = x^{n+2} + R_{n+2}x - BR_{n+1}$ have only one change of sign, thus according to the Descartes' rule of signs, the polynomial $g_n(x)$ has exactly one negative real zero. But $g_n(x) = (x^2 - Ax - B)q_n(x)$ implies that $g_n(\beta) = 0$, where $\beta < 0$, and so the polynomial $q_n(x)$ can not have any negative real zero.

b) Since $n \geqslant 3$ is odd, thus the existence of at least one negative real zero is obvious. We have only to prove that exactly one negative real zero exists. The polynomial

$$g_n(-x) = (-x)^{n+2} - R_{n+2}(-x) - BR_{n+1} = -x^{n+2} + R_{n+2}x - BR_{n+1}$$

shows that among its coefficients there are two changes of signs, thus according to the Descartes' rule of signs, the polynomial $g_n(x)$ has either two negative real zeros or no one. But $g_n(x) = (x^2 - Ax - B)q_n(x)$ implies that for $\beta < 0$ $g_n(\beta) = 0$. Although, $g_n(\alpha) = 0$ also holds, but $\alpha > 0$. That is, an other negative real zero of $g_n(x)$ must exist. Because of $g_n(x) = (x^2 - Ax - B)q_n(x)$ this zero must be the zero of the polynomial $q_n(x)$.

This terminated the proof of the theorem. □

In this part of the paper we deal with the localization of the zeros of the polynomials $q_n(x)$ defined in (1.4).

Theorem 3.2. *Let $z \in \mathbf{C}$ denote an arbitrary zero of the polynomial $q_n(x)$. For $n \geqslant 1$*

$$A \leqslant |z| \leqslant A + \frac{B}{A},$$

where A and B are positive integers from (1.1).

Proof. To apply Lemma 2.2 for the polynomial $q_n(x)$ we have to determine the minimal and maximal values in the sequence

$$\frac{R_{n+1}}{R_n}, \frac{R_n}{R_{n-1}}, \dots, \frac{R_2}{R_1}.$$

According to Lemma 2.3, these are $\frac{R_2}{R_1}$ and $\frac{R_3}{R_2}$, respectively. But by (1.1), $\frac{R_2}{R_1} = A$ and $\frac{R_3}{R_2} = \frac{A^2+B}{A} = A + \frac{B}{A}$, which match the statement of the theorem. □

Remarks. 1) If $n \geqslant 3$ and is odd then for the only one negative real zero z_n of the polynomial $q_n(x)$

$$-A - \frac{B}{A} \leqslant z_n \leqslant -A. \tag{3.1}$$

2) If we know the exact value of A and B then the estimation in (3.1) can be improved. E.g. in the case of the Fibonacci-sequence ($A = B = 1$) (see in [1])

$$-2 < -\frac{1+\sqrt{5}}{2} < z_n \leqslant -1.$$

3) For arbitrary zero $z^\star$ of the polynomial $q_n^\star(x)$ (1.5)

$$\frac{1}{A + \frac{B}{A}} \leqslant |z^\star| \leqslant \frac{1}{A}$$

holds.

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