Annales Mathematicae et Informaticae
34 (2007) pp. 17–28
http://www.ektf.hu/tanszek/matematika/ami

The remainder term in Fourier series and its relationship with the Basel problem

V. Barrera-Figueroa[a], A. Lucas-Bravo[a], J. López-Bonilla[b]

[a] Unidad Profesional Interdisciplinaria de Ingeniería y Tecnologías Avanzadas,
Departamento de Telemática
e-mail: vbarreraf@ipn.mx, alucasb@ipn.mx

[b] Instituto Politécnico Nacional, Escuela Superior de Ingeniería Mecánica y Eléctrica,
Sección de Estudios de Postgrado e Investigación
e-mail: jlopezb@ipn.mx

Submitted 5 September 2007; Accepted 20 November 2007

Abstract

In this paper it is shown several approximation formulae for the remainder term of the Fourier series for a wide class of functions satisfying specific boundary conditions. Also it is shown that the remainder term is related with the Basel problem and the Riemann zeta function, which can be interpreted as the energy of discrete-time signals; from this point of view, their energy can be calculated with a direct formula instead of an infinite series. The validity of this algorithm is established by means several proofs.

Keywords: Fourier series remainder term, discrete-time signal, Basel problem, slow varying-type series.

1. Introduction

Fourier series is a mathematical tool for characterizing the frequency content of a periodic signal which satisfies the Dirichlet conditions [1]. However, the Fourier series is frequently applied to non-periodic functions, made artificially periodic by extending periodically its original domain. In practice, with a sufficiently large number of terms, a finite expansion can be built upon the Fourier series for representing accurately enough the function. Such finite representation carries implicitly a remainder term which must be estimated [2].

The calculation of the remainder term is expressed via a mean square error between the infinite series and the finite expansion, which provides us an enclosed range of values where the error can be found, instead of an exact formula. Such estimations stir up the appearance of slow varying-type series, as in the Basel problem series, which in general are expressed by the Riemann zeta function. This let us establish a relation between it and a discrete-time signal, usually defined for all the natural numbers. Therefore the approximation of the remainder term in Fourier series can be employed as an excellent way for calculating the energy of a discrete-time signal. The energy calculation embraces a small number of terms instead of an infinite number, which brings us accurately enough results whose validity is proved in this paper.

2. Integral approach of slow-varying series

In the calculation of the remainder term in finite Fourier expansion, appears series whose members are expressed as the product of a periodic term and a function which varies slowly between successive values of. This let us get a very good approximation of the series [5]:

$$\sum_{k=1}^{\infty} e^{jk\alpha}\varphi(k). \tag{2.1}$$

Let us integrate the k^{th} term around $k-1/2$ and $k+1/2$:

$$\int_{k-1/2}^{k+1/2} \varphi(\xi)e^{j\xi\alpha}d\xi = \int_{-1/2}^{1/2} \varphi(k+t)e^{j(k+t)\alpha}dt$$

$$= \frac{1}{j\alpha}\varphi(k+t)e^{j(k+t)\alpha}\Bigg|_{-1/2}^{1/2} - \frac{1}{j\alpha}\int_{-1/2}^{1/2} \varphi'(k+t)e^{j(k+t)\alpha}dt. \tag{2.2}$$

However, since $\varphi'(k+t)$ tends to zero asymptotically, it is possible to establish the following approximation:

$$\int_{-1/2}^{1/2} \varphi(k+t)e^{j(k+t)\alpha}dt \approx \frac{e^{jk\alpha}}{j\alpha}[\varphi(k+1/2)e^{j\alpha/2} - \varphi(k-1/2)e^{-j\alpha/2}]. \tag{2.3}$$

Because of the slow variation of $\varphi(k)$ we have that $\varphi(k+1/2) \approx \ \varphi(k-1/2) \approx \varphi(k)$, therefore the integral is:

$$\int_{-1/2}^{1/2} \varphi(k+t)e^{j(k+t)\alpha}dt \approx e^{jk\alpha}\varphi(k)\frac{\sin\alpha/2}{\alpha/2}. \tag{2.4}$$

By changing the integrating variable we have:

$$e^{jk\alpha}\varphi(k) \approx \frac{\alpha}{2\sin\alpha/2}\int_{k-1/2}^{k+1/2} \varphi(\xi)e^{j\xi\alpha}d\xi, \tag{2.5}$$

which transforms the original series into a series of integrals:

$$\sum_{k=1}^{\infty} e^{jk\alpha}\varphi(k) \approx \frac{\alpha}{2\sin\alpha/2}\sum_{k=1}^{\infty}\int_{k-1/2}^{k+1/2}\varphi(\xi)e^{j\xi\alpha}d\xi. \tag{2.6}$$

Since the integration limits are contiguous, the series becomes in just one integral:

$$\sum_{k=1}^{\infty} e^{jk\alpha}\varphi(k) \approx \frac{\alpha}{2\sin\alpha/2}\int_{1/2}^{\infty}\varphi(\xi)e^{j\xi\alpha}d\xi. \tag{2.7}$$

3. The Zeta function as the generalization of the Basel problem

The Basel problem is a famous issue in the Number Theory because of its ingenious solution provided by Leonhard Euler, and its relationship to the distribution of the prime numbers. The problem consists in calculating the exact sum of the following series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right). \tag{3.1}$$

Euler's method uses the Taylor series for the sine function, which is a polynomial whose roots are $x = k\pi$, $k \in \mathbb{Z}$. Thus, with the Fundamental Theorem of Algebra, the polynomial $\sin x/x$ can be written in terms of its roots [3]:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots = A(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\cdots, \tag{3.2}$$

where A is a proportionality constant. Since each factor has the form $x^2 - n^2\pi^2 = 0$, they can be expressed as $1 - x^2/n^2\pi^2$, transforming the polynomial into:

$$\frac{\sin x}{x} = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots, \tag{3.3}$$

by multiplying all the factors and gathering the coefficients belonging to x^2, results the series:

$$-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}\cdots = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}, \tag{3.4}$$

From (3.2) we get the coefficient of x^2 as $-1/3!$, therefore:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3.5}$$

The same procedure, after being applied in the other resulting powers of the multiplication (3.3), gives a set of impressive series, all of which are based in even powers:

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\quad \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450},\quad \sum_{n=1}^{\infty}\frac{1}{n^{10}}=\frac{\pi^{10}}{93555},\quad \dots \tag{3.6}$$

The generalization of the Basel problem for real powers is gotten by the Riemann zeta function, defined as [6]:

$$\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x},\quad x\neq 1. \tag{3.7}$$

The case $x=1$ is avoided since the series becomes divergent, Figure 1. For even powers the function gives exact values, proportional to even powers of π, as shown in (3.6); for odd powers it is not possible to get such an exact representations. The

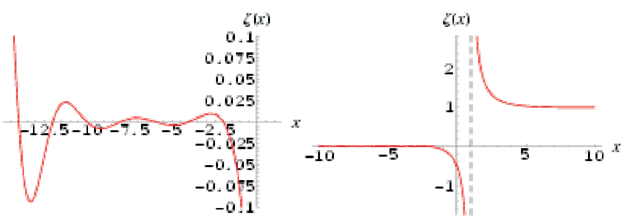

Figure 1: Plot of the Riemann zeta function.

Bernoulli numbers B_n are a set of rational numbers defined by the series [4]:

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!},$$
$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad \dots \tag{3.8}$$

The zeta function is related with them for integer values of the argument x as:

$$\zeta(n)=\frac{2^{n-1}|B_n|\pi^n}{n!},\; B_n=(-1)^{n+1}n\zeta(1-n),\quad n\in\mathbb{N}. \tag{3.9}$$

4. The remainder term of Fourier series

The Fourier series develops a function by means of an infinite series of trigonometric terms; its convergence is assured by Dirichlet conditions. However, in practice, it is not possible to take an infinite number of such orthogonal functions, but a finite number of them for performing a finite Fourier expansion $f_n(x)$, formed by n terms. Fourier series convergence shows that by taking a sufficiently large number of terms, the difference between $f(x)$ and $f_n(x)$, named the remainder term, can be made as small as we desire:

$$\eta_n(x) = f(x) - f_n(x). \tag{4.1}$$

Let us suppose that $f^{(m)}(x)$ exists, although its continuity is not demanded; however, the continuity of $f(x), f'(x), f''(x), \ldots, f^{m-1}(x)$ is required for setting the following boundary conditions:

$$f(\pi) = f(-\pi),\ f'(\pi) = f'(-\pi), \ldots, f^{m-1}(\pi) = f^{m-1}(-\pi). \tag{4.2}$$

The existence of the above conditions let us simplify the integration of the coefficients in Fourier series, performed by parts successively m times. They can be gathered in a complex coefficient:

$$a_k + jb_k = \frac{j^m}{\pi k^m} \int_{-\pi}^{\pi} f^{(m)}(\xi) e^{jk\xi} d\xi, \tag{4.3}$$

where the Fourier series is the real part of the series:

$$f(x) = \sum_{k=1}^{\infty} (a_k + jb_k) e^{-jkx} = \int_{-\pi}^{\pi} f^{(m)}(\xi) \left[\frac{j^m}{\pi} \sum_{k=1}^{\infty} \frac{e^{jk(\xi - x)}}{k^m} \right] d\xi. \tag{4.4}$$

The index $k = 0$ has been omitted since $f(x)$ stands for $f(x) - a_0/2$. Let us change the integrating variable by $\theta = \xi - x$, therefore, $f(x)$ is written in terms of the kernel-type series $G_m(\theta)$:

$$f(x) = \int_{-\pi}^{\pi} f^{(m)}(\theta + x) G_m(\theta) d\theta, \quad G_m(\theta) = \frac{j^m}{\pi} \sum_{k=1}^{\infty} \frac{e^{jk\theta}}{k^m}. \tag{4.5}$$

In the finite expansion $f_n(x)$, the kernel-type series must add only n terms, thus the remainder term is expressed in function of another kernel-type series $g_n^m(\theta)$:

$$\eta_n(x) = \int_{-\pi}^{\pi} f^{(m)}(\theta + x) g_n^m(\theta) d\theta, \quad g_n^m(\theta) = \frac{j^m}{\pi} \sum_{k=n+1}^{\infty} \frac{e^{jk\theta}}{k^m}. \tag{4.6}$$

The simpler method for getting the remainder term is based on Cauchy inequality:

$$\left[\int_a^b f(x) g(x) dx \right]^2 \leqslant \int_a^b f^2(x) dx \int_a^b g^2(x) dx. \tag{4.7}$$

After applied it in (4.6) we get:

$$\eta_n^2(x) \leqslant \int_{-\pi}^{\pi} f^{(m)^2}(\theta + x) d\theta \int_{-\pi}^{\pi} [g_n^m(\theta)]^2 d\theta. \tag{4.8}$$

In this case, we can take advantage of the orthogonality of the members of the series $g_n^m(\theta)$, by taking their real part. The integral of the square of kernel-type series is:

$$\int_{-\pi}^{\pi}[g_n^m(\theta)]^2 d\theta = \frac{1}{\pi^2}\sum_{k=n+1}^{\infty}\int_{-\pi}^{\pi}\frac{\cos k\theta}{k^m}\sum_{l=n+1}^{\infty}\frac{\cos l\theta}{l^m}d\theta = \frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}}. \tag{4.9}$$

The above series seems to be related with the Riemann zeta function, however, we cannot get an exact result since the series starts from $n+1$. For estimation purposes, we can use the integral approach of a slow varying-type series, whose periodic part is the unitary function, i.e., $\alpha = 0$. The slow varying part is the function $\varphi(k) = 1/k^{2m}$, which varies slowly, since m and n are supposed to be great:

$$\frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}} \approx \frac{1}{\pi}\int_{n+1/2}^{\infty}\frac{d\xi}{\xi^{2m}} = \frac{1}{\pi(2m-1)(n+1/2)^{2m-1}}. \tag{4.10}$$

The integral of ${f^{(m)}}^2$, should be identified as the norm of the m^{th} derivative of $f(x)$, represented by N_m^2, therefore the remainder term is bounded by:

$$|\eta_n(x)| < \frac{N_m}{\sqrt{\pi(2m-1)}(n+1/2)^{m-1/2}}. \tag{4.11}$$

Another method for getting the remainder term is by evaluating reliably the kernel-type series $g_n^m(\theta)$ with the integral approach of a slow varying-type series, where the slow varying function corresponds with $\varphi(k) = 1/k^m$. With the exception of small values around $\theta = 0$, we can use the asymptotic behavior of the integral:

$$g_n^m(\theta) \approx \frac{j^m\theta}{2\pi\sin\theta/2}\int_{n+1/2}^{\infty}\frac{e^{j\xi\theta}}{\xi^m}d\xi \approx \frac{j^{m+1}}{2\pi\sin\theta/2}\frac{e^{j(n+1/2)\theta}}{(n+1/2)^m}. \tag{4.12}$$

For estimation purposes, the remainder term can be calculated by means the following inequality:

$$|\eta_n(x)| \leqslant \int_{-\pi}^{\pi}|f^{(m)}(\theta+x)||g_n^m(\theta)|d\theta = |f^{(m)}(x)|_{max}\int_{-\pi}^{\pi}|g_n^m(\theta)|d\theta. \tag{4.13}$$

After taking the real part of $g_n^m(\theta)$ and integrating it, we get the following formula:

$$|\eta_n(x)| < \frac{2}{(n+1/2)^{m-1}}\frac{\ln(n+1/2)\pi}{(n+1/2)\pi}|f^m(x)|_{max}. \tag{4.14}$$

5. Mean square error in Fourier series

Frequently the remainder term is known as the error term, for its interpretation is obvious. However, it is more suitable to handle a mean square error for practical issues:

$$\eta^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\eta_n^2(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-f_n(x)]^2 dx. \tag{5.1}$$

The orthogonality properties let us express the mean square error in function of the coefficients in Fourier series:

$$\eta^2 = \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) - \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2) = \frac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2 + b_k^2), \tag{5.2}$$

where the mean square error results equal to the square of the remainder term:

$$\eta^2 = \frac{\eta_n^2}{2\pi}\int_{-\pi}^{\pi} dx = \eta_n^2. \tag{5.3}$$

The above formulae let us find out a relation between the norm of the m^{th} derivative of $f(x)$ and the coefficients of its Fourier series. By substituting (4.9) into (4.8) we have:

$$\eta_n^2 \leqslant \frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}}\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi, \tag{5.4}$$

from which we get the following inequality:

$$\frac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2 + b_k^2) \leqslant \frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^{2m}}\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi, \tag{5.5}$$

which provides us the wanted relation:

$$a_k^2 + b_k^2 < \frac{1}{\pi k^{2m}}\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi. \tag{5.6}$$

If we consider that in the inequality (5.5) both series start from $k = 1$, we get:

$$\frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leqslant \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi, \tag{5.7}$$

where the left side is proportional to the integral of $f^2(x)$:

$$\sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx, \tag{5.8}$$

from which the following inequality is gotten:

$$\frac{1}{2}\int_{-\pi}^{\pi} f^2(x)dx \leqslant \sum_{k=1}^{\infty}\frac{1}{k^{2m}}\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi. \tag{5.9}$$

This series is expressed in terms of the zeta function, from which results the following impressive inequality:

$$\int_{-\pi}^{\pi} f^2(x)dx \leqslant \frac{(2\pi)^{2m}}{(2m)!}|B_{2m}|\int_{-\pi}^{\pi} f^{(m)^2}(\xi)d\xi. \tag{5.10}$$

6. Energy of discrete-time signals

A discrete-time signal $x(k)$ is a single value function defined at discrete points of the domain, which represents the samples of a continuous-time function $x_a(t)$, related with the first one by:

$$x(k) = x_a(kT), \quad k \in \mathbb{Z}, \tag{6.1}$$

being T the sampling rate. For discrete-time signals we can define their energy E as that dissipated by a unitary resistance:

$$E = \sum_{k=-\infty}^{\infty} |x(k)|^2. \tag{6.2}$$

For energy signals, the above series converges. However, if the series diverges, the function is said to be a power signal [7]. In general, power signals are periodic functions, where their mean power P, measured in a complete period N, converges:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |x(k)|^2. \tag{6.3}$$

In practice, it is not possible to perform an infinite summation for calculating the energy of a discrete-time signal, since with a representative number of terms we can get an approximation of the series, for the upper terms can be neglected since energy signals show a decreasing behavior; therefore we have the following approximation:

$$E_n = \sum_{k=1}^{n} |x(k)|^2, \tag{6.4}$$

where $x(k)$ is supposed to be a causal signal, i.e., $x(k) = 0$ for $k \leqslant 0$. Therefore, the Riemann zeta function, for even arguments, gives the exact value of the energy of a discrete-time signal:

$$E = \zeta(2m) = \sum_{k=1}^{\infty} \frac{1}{k^{2m}}, \quad m \in \mathbb{N}. \tag{6.5}$$

which is written as a sequence of weighted unitary impulse:

$$x(n) = \sum_{k=1}^{\infty} \frac{\delta(n-k)}{k^m}, \quad \delta(n-k) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \tag{6.6}$$

The approximation of the energy of the signal is written in terms of its total energy, expressed by the zeta function:

$$E_n = \sum_{k=1}^{n} \frac{1}{k^{2m}} = \zeta(2m) - \sum_{k=n+1}^{\infty} \frac{1}{k^{2m}}. \tag{6.7}$$

In this formula, we can use the integral approach of a slow varying-type series since the second one varies slowly, as is required. Therefore:

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{2m}} \approx \int_{n+1/2}^{\infty} \frac{d\xi}{\xi^{2m}} = \frac{1}{(2m-1)} \frac{1}{(n+1/2)^{2m-1}}. \tag{6.8}$$

Hence, the next formula has the advantage of bring us a very accurate value of the energy of the discrete-time signal without developing the sum until the n^{th} term:

$$E_n \approx \zeta(2m) - \frac{1}{(2m-1)} \frac{1}{(n+1/2)^{2m-1}}. \tag{6.9}$$

The following tables present a comparative analysis which demonstrates the validity of (6.9) as a reliable approximation formula for the finite expansion (6.4). For doing so, we must programming the formula (6.7) by using double precision floating point variables, defined in C++ language like of double type. Figure 2 shows the flow diagram of the main program.

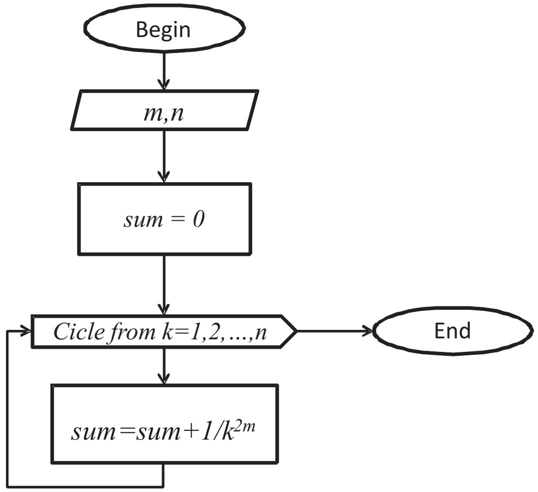

Figure 2: Algorithm for performing the expansion E_n.

7. Conclusions

The formulae used for getting the bounded values of the remainder terms were deduce from the integral approach of a slow varying-type series, which let us calculate the remainder term without performing the infinite sum of the series. In fact, in this work has been proved that the remainder term can be related with the Riemann zeta function, which aroused from the Basel problem.

n	E_n summation	E_n approximation	Absolute Difference
1	1.000000000000000000	0.978267400181559776	0.021732599818440224
10	1.549767731166540760	1.549695971610131060	0.000071759556409701
100	1.634983900184892260	1.634983818092007550	0.000000082092884712
1000	1.643934566681561240	1.643934566598351350	0.000000000083209883
10000	1.644834071848064960	1.644834071847976360	0.000000000000088596
100000	1.644924066898243000	1.644924066898226120	0.000000000000016875

Table 1: Case $m = 1$.

n	E_n summation	E_n approximation	Absolute Difference
1	1.000000000000000000	0.983557801612372495	0.016442198387627505
10	1.082036583493756640	1.082035287844960840	0.000001295648795807
100	1.082322905344472730	1.082322905328218180	0.000000000016254553
1000	1.082323233378305940	1.082323233378304160	0.000000000000001776
10000	1.082323233710861480	1.082323233710804630	0.000000000000056843
100000	1.082323233710861480	1.082323233711137480	0.000000000000276001

Table 2: Case $m = 2$.

n	E_n summation	E_n approximation	Absolute Difference
1	1.000000000000000000	0.991005613424777998	0.008994386575222002
10	1.017341512441431340	1.017341494932115790	0.000000017509315553
100	1.017343061964943730	1.017343061964941290	0.000000000000002442
1000	1.017343061984441020	1.017343061984448570	0.000000000000007550
10000	1.017343061984441020	1.017343061984448790	0.000000000000007772
100000	1.017343061984441020	1.017343061984448790	0.000000000000007772

Table 3: Case $m = 3$.

The Riemann zeta function can be parsed as the energy of a discrete-time energy signal. For calculating accurately its total energy, it is not necessary to perform a large expansion of terms, but to use a formula which is gotten from the study of the remainder term of the Fourier series.

As can be seen from Tables 1– 5, the results demonstrate the virtue of using the formula (6.9) instead of counting n terms. Even if the expansion is formed by only one term, the error involved is in the order of 0.1% for $m = 5$, and 2.1% for $m = 1$. In addition, from the tables we can assure the convergence of the results by taking only ten terms in all of the cases; by taking a large number of terms, the results show that occur a kind of saturation in the results of the program, since there

n	E_n summation	E_n approximation	Absolute Difference
1	1.000000000000000000	0.995716261417096016	0.004283738582903984
10	1.004077346255262570	1.004077346045353590	0.000000000209908979
100	1.004077356197943030	1.004077356197942580	0.000000000000000444
1000	1.004077356197943030	1.004077356197943920	0.000000000000000888
10000	1.004077356197943030	1.004077356197943920	0.000000000000000888
100000	1.004077356197943030	1.004077356197943920	0.000000000000000888

Table 4: Case $m = 4$.

n	E_n summation	E_n approximation	Absolute Difference
1	1.000000000000000000	0.998104320141845691	0.001895679858154309
10	1.000994575058549610	1.000994575056194600	0.000000000002355005
100	1.000994575127818200	1.000994575127817750	0.000000000000000444
1000	1.000994575127818200	1.000994575127817750	0.000000000000000444
10000	1.000994575127818200	1.000994575127817750	0.000000000000000444
100000	1.000994575127818200	1.000994575127817750	0.000000000000000444

Table 5: Case $m = 5$.

exist no variations in the calculations while increasing the number of summands. This can be interpreted as that the first elements have more energy than the upper ones. Therefore, the use of a finite expansion for calculating the energy of the discrete-time signal is justified, since the upper terms can be neglected.

References

[1] Cantor, G., Contributions to the Founding of the Theory of Transfinite Numbers *Dover Publications, Inc.* (1995), 1–82.

[2] Fejér, L., Untersuchen Über Fouriersche Reihen, *Math. Annalen*, Vol. 58 (1904), 51–69.

[3] Kimble, G., Euler's Other Proof, *Mathematics Magazine*, Vol. 60 (1987), 282.

[4] Lanczos, C., Discourse on Fourier Series, *Oliver & Boyd, Edinburgh*, (1996), 45–75, 109.

[5] Lanczos, C., Linear Differential Operators, *Dover Publications, Inc.* (1997), 49–99.

[6] Penrose, R., The Road to Reality, *Jonathan Cape*, (2004), 211.

[7] Proakis, J.G., Manolakis, D.G., Digital Signal Processing. Principles, Algorithms and Applications, *Prentice Hall* (1999), 43–52.

V. Barrera-Figueroa and A. Lucas-Bravo
Av. I.P.N. No. 2580 Col. Barrio La Laguna Ticomán, CP 07340, México D.F.

J. López-Bonilla
UPALM Edif. Z-4, 3er piso, Col. Lindavista CP 07738, México D.F.