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Some properties of solutions of systems of neutral differential equations

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Abstract

The aim of this paper is to present some sufficient conditions for the oscillatory and asymptotic properties of solutions of the system of differential equations of neutral type.

Keywords: neutral equation, oscillatory solution

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1. Introduction

In this paper we consider three-dimensional systems of neutral differential equations of the form:

$$\begin{aligned} \big(y_1(t) - p\, y_1(t-\tau)\big)' &= p_1(t)\, f_1(y_2(h_2(t))), \\ y_2'(t) &= p_2(t)\, f_2(y_3(h_3(t))), \\ y_3'(t) &= \sigma\, p_3(t)\, f_3(y_1(h_1(t))), \end{aligned} \tag{1.1}$$

where $t \in R_+ = [0,\infty)$, $\sigma = 1$ or $\sigma = -1$ and the following conditions are assumed to hold without further mention:

(a) $\tau > 0$, $0 < p < 1$;

(b) $p_i \in C(R_+, R_+)$, $i = 1,2,3$ are not identically zero on any subinterval $[T,\infty) \subset R_+$ and

$$\int^{\infty} p_j(t)\, dt = \infty \qquad \text{for } j = 1,2;$$

(c) $h_i \in C(R_+, R)$ and

$$\lim_{t\to\infty} h_i(t) = \infty \qquad \text{for } i = 1, 2, 3;$$

(d) $f_i(u) = |u|^{\alpha_i} \operatorname{sgn} u$ where $\alpha_i \in R$, $\alpha_i > 0$, $i = 1, 2, 3$.

The assumption (d) implies that

(e) $u f_i(u) > 0$ for $u \neq 0$ and $f_i \in C(R, R)$, $i = 1, 2, 3$ are nondecreasing functions.

Surveying the rapidly expanding literature devoted to the study of oscillatory and asymptotic properties of neutral differential equations, one finds that few papers concern systems of neutral equations (for example [1-9]). The purpose of this paper is to establish some criteria for the oscillation of the system (1.1) for the following cases

I) $\sigma = -1$ and $0 < \alpha_1\,\alpha_2\,\alpha_3 < 1$;

II) $\sigma = -1$ and $\alpha_1\,\alpha_2\,\alpha_3 = 1$;

III) $\sigma = 1$.

Another cases (for example $\sigma = -1$ and $\alpha_1 \geqslant 1$, $0 < \alpha_2 \leqslant 1$, $\alpha_3 > 1$) are studied in [6]. Theorem 1 and Theorem 2 are generalizations of results of V. N. Shevelo, N. V. Varech, A. G. Gritsai in paper [7].

For any $y_1(t)$ we define $z(t)$ by

$$z(t) = y_1(t) - p\, y_1(t - \tau).$$

Let $t_0 \geqslant 0$ be such that

$$t_1 = \min\left\{ t_0 - \tau,\ \inf_{t \geqslant t_0} h_i(t),\ i = 1, 2, 3 \right\} \geqslant 0.$$

A vector function $y = (y_1,\, y_2,\, y_3)$ is a solution of the system (1.1) if there exists a $t_0 \geqslant 0$ such that y is continuous on $[t_1, \infty)$, $z(t)$, $y_2(t)$, $y_3(t)$ are continuously differentiable on $[t_0, \infty)$ and y satisfies system (1.1) on $[t_0, \infty)$.

Denote by W the set of all solutions $y = (y_1,\, y_2,\, y_3)$ of the system (1.1) which exist on some ray $[T_y, \infty) \subset R_+$ and satisfy

$$\sup\left\{ \sum_{i=1}^{3} |\, y_i(t)| \;:\; t \geqslant T \right\} > 0 \qquad \text{for any } T \geqslant T_y.$$

Such a solution is called a proper solution. A proper solution $y \in W$ is defined to be nonoscillatory if there exists a $T_y \geqslant 0$ such that its every component is different from zero for all $t \geqslant T_y$. Otherwise a proper solution $y \in W$ is defined to be oscillatory.

2. Some basic lemmas

We begin with some lemmas which will be useful in the sequel.

Lemma 2.1. ([2, Lemma1]) *Let (a)–(d) hold and $y \in W$ be a nonoscillatory solution of* (1.1). *Then there exists a $t_0 \geqslant 0$ such that $z(t)$, $y_2(t)$, $y_3(t)$ are monotone functions of constant sign on the interval $[t_0, \infty)$.*

Let $y = (y_1, y_2, y_3) \in W$ be a nonoscillatory solution of (1.1). Taking into account the Lemma 2.1 we obtain:

$$y_1(t)\, z(t) > 0 \qquad \text{for } t \geqslant t_0 \tag{2.1}$$

or

$$y_1(t)\, z(t) < 0 \qquad \text{for } t \geqslant t_0. \tag{2.2}$$

Denote by N^+ (or N^-) the set of components $y_1(t)$ of all nonoscillatory solutions y of system (1.1) such that (2.1) (or (2.2)) is satisfied.

For the components $y_1(t)$ of the nonoscillatory solutions hold the following lemmas.

Lemma 2.2. ([5, Lemma3]) *Let (a) hold and $y_1(t) \in N^-$. Then $\lim\limits_{t\to\infty} y_1(t) = 0$, $\lim\limits_{t\to\infty} z(t) = 0$.*

Lemma 2.3. ([3, Lemma2]) *Let (a) hold and $y_1(t) \in N^+$. If $\lim\limits_{t\to\infty} z(t) = 0$, then $\lim\limits_{t\to\infty} y_1(t) = 0$.*

3. Oscillation theorems

Theorem 3.1. *Assume that $\sigma = -1$ and*

(A1) $h_3(h_2(h_1(t))) \leqslant t$, *$h_i(t)$ are nondecreasing functions for $i = 2, 3$;*

(A2) $0 < \alpha_1\, \alpha_2\, \alpha_3 < 1$.

If

(A3)

$$\int^{\infty} p_3(v)\Big[\int_0^{h_1(v)} p_1(u)\Big(\int_0^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du\Big]^{\alpha_3} dv = \infty,$$

(A4)

$$\int^{\infty} p_2(t)\Big(\int_{h_3(t)}^{\infty} p_3(s)ds\Big)^{\alpha_2} dt = \infty,$$

then every proper solution $y \in W$ of (1.1) *is either oscillatory or $y_i(t)$, i=1,2,3 tend monotonically to zero as $t \to \infty$.*

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (1.1). According to Lemma 2.1 there exists a $t_0 \geqslant 0$ such that $z(t)$, $y_2(t)$, $y_3(t)$ are monotone functions of constant sign on the interval $[t_0, \infty)$. Without loss of generality we may assume that $y_1(t) > 0$ for $t \geqslant t_0$. Then either $y_1(t) \in N^+$ or $y_1(t) \in N^-$ for $t \geqslant t_0$.

I. Let $y_1(t) \in N^+, t \geqslant t_0$. Then $z(t) > 0$, $t \geqslant t_0$ and using the assumptions (c), (d) and (b), the third equation of (1.1) implies that $y_3(t)$ is a decreasing function for $t \geqslant t_1 \geqslant t_0$.

I.1 Let $y_3(t) < 0$, $t \geqslant t_2 \geqslant t_1$. In regard of (c) there exists a $t_3 \geqslant t_2$ such that $y_3(h_3(t)) < 0$ for $t \geqslant t_3$. The assumptions (d), (b) and the second equation of (1.1) imply that $y_2(t)$ is a decreasing function for $t \geqslant t_3$.

In view of (c) there exists a $t_4 \geqslant t_3$ such that $h_3(t) \geqslant t_3$ for $t \geqslant t_4$. Using the monotonicity of $y_3(t)$ we have $y_3(h_3(t)) \leqslant y_3(t_3)$ and hence $| \, y_3(h_3(t)) \, | \geqslant K_1$, where $K_1 = -y_3(t_3) > 0$ for $t \geqslant t_4$. Raising this inequality to the power of α_2 and multiplying by $-p_2(t)$ the second equation of (1.1) implies

$$y_2'(t) \leqslant -K_1^{\alpha_2} \, p_2(t), \quad t \geqslant t_4. \tag{3.1}$$

Integrating (3.1) from t_4 to t and in regard of (b) we obtain $\lim\limits_{t\to\infty} y_2(t) = -\infty$. Therefore $y_2(t) < 0$ for $t \geqslant t_5 \geqslant t_4$.

In view of (c) there exists a $t_6 \geqslant t_5$ such that $h_2(t) \geqslant t_5$ for $t \geqslant t_6$. Using the monotonicity of $y_2(t)$ we have $y_2(h_2(t)) \leqslant y_2(t_5)$ and hence $| \, y_2(h_2(t)) \, | \geqslant K_2$, where $K_2 = -y_2(t_5) > 0$, $t \geqslant t_6$. Raising the last inequality to the power of α_1 and multiplying by $-p_1(t)$ the first equation of (1.1) implies

$$z'(t) \leqslant -K_2^{\alpha_1} \, p_1(t), \quad t \geqslant t_6. \tag{3.2}$$

Integrating (3.2) from t_6 to t and in regard of (b) we obtain $\lim\limits_{t\to\infty} z(t) = -\infty$. Therefore $z(t) < 0$ for $t \geqslant t_7 \geqslant t_6$ which is a contradiction with positivity of $z(t)$ for $t \geqslant t_0$.

I.2 Assume that $y_3(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. In view of (c) there exists a $t_3 \geqslant t_2$ such that $y_3(h_3(t)) > 0$ for $t \geqslant t_3$. The assumptions (d), (b) and the second equation of (1.1) imply that $y_2(t)$ is an increasing function for $t \geqslant t_3$.

I.2.a Let $y_2(t) > 0$ for $t \geqslant t_4 \geqslant t_3$. Integrating the second equation of (1.1) from t_4 to t we obtain

$$y_2(t) \geqslant y_2(t) - y_2(t_4) = \int_{t_4}^{t} \Big(y_3(h_3(s)) \Big)^{\alpha_2} p_2(s) \, ds, \quad t \geqslant t_4. \tag{3.3}$$

In regard of monotonicity of functions $h_3(t)$, $y_3(t)$ the inequality $t_4 \leqslant s \leqslant t$ may be rewritten as $\Big(y_3(h_3(t_4))\Big)^{\alpha_2} \geqslant \Big(y_3(h_3(s))\Big)^{\alpha_2} \geqslant \Big(y_3(h_3(t))\Big)^{\alpha_2}$. Then from (3.3) we get

$$y_2(t) \geqslant \Big(y_3(h_3(t))\Big)^{\alpha_2} \int_{t_4}^{t} p_2(s)\,ds, \quad t \geqslant t_4.$$

In view of (c) there exists a $t_5 \geqslant t_4$ such that $h_2(t) \geqslant t_4$ for $t \geqslant t_5$. Then the last inequality holds for $h_2(t)$, $t \geqslant t_5$, too:

$$y_2(h_2(t)) \geqslant \Big(y_3(h_3(h_2(t)))\Big)^{\alpha_2} \int_{t_4}^{h_2(t)} p_2(s)\,ds, \quad t \geqslant t_5. \tag{3.4}$$

Raising (3.4) to the power of α_1 and multiplying by $p_1(t)$ the first equation of (1.1) implies:

$$z'(t) \geqslant p_1(t)\Big(y_3(h_3(h_2(t)))\Big)^{\alpha_1\alpha_2} \Big(\int_{t_4}^{h_2(t)} p_2(s)\,ds\Big)^{\alpha_1}, \quad t \geqslant t_5.$$

Integrating this inequality from t_5 to t and using the inequality $y_1(t) \geqslant z(t) \geqslant z(t) - z(t_5)$ we have

$$y_1(t) \geqslant \int_{t_5}^{t} p_1(u)\Big(y_3(h_3(h_2(u)))\Big)^{\alpha_1\alpha_2} \Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du, \quad t \geqslant t_5. \tag{3.5}$$

In regard of monotonicity of functions $h_2(t)$, $h_3(t)$ and $y_3(t)$ the inequality $t_5 \leqslant u \leqslant t$ may be rewritten as

$$\Big(y_3(h_3(h_2(u)))\Big)^{\alpha_1\alpha_2} \geqslant \Big(y_3(h_3(h_2(t)))\Big)^{\alpha_1\alpha_2} \text{for } t \geqslant t_5.$$

Combining the last inequality and (3.5) we obtain

$$y_1(t) \geqslant \Big(y_3(h_3(h_2(t)))\Big)^{\alpha_1\alpha_2} \int_{t_5}^{t} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du, \quad t \geqslant t_5. \tag{3.6}$$

In view of (c) there exists a $t_6 \geqslant t_5$ such that $h_1(t) \geqslant t_5$ for $t \geqslant t_6$. Then (3.6) holds for $h_1(t)$, $t \geqslant t_6$, too and raising to the power of α_3 we get

$$\Big(y_1(h_1(t))\Big)^{\alpha_3} \geqslant \Big(y_3(h_3(h_2(h_1(t))))\Big)^{\alpha_1\alpha_2\alpha_3} \Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{\alpha_3} \tag{3.7}$$

for $t \geqslant t_6$. Multiplying (3.7) by $-p_3(t)$ and using the third equation of system (1.1) we have

$$y_3'(t) \leqslant -p_3(t)\Big(y_3(h_3(h_2(h_1(t))))\Big)^{\alpha_1\alpha_2\alpha_3}\Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{\alpha_3} \tag{3.8}$$

for $t \geqslant t_6$. Taking into account (A1) and the monotonicity of $y_3(t)$ we obtain

$$\Big(y_3(h_3(h_2(h_1(t))))\Big)^{\alpha_1\alpha_2\alpha_3} \geqslant (y_3(t))^{\alpha_1\alpha_2\alpha_3} \qquad \text{for } t \geqslant t_6.$$

Therefore (3.8) may be rewritten as

$$\frac{y_3'(t)}{(y_3(t))^{\alpha_1\alpha_2\alpha_3}} \leqslant -p_3(t)\Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{\alpha_3}, \quad t \geqslant t_6. \tag{3.9}$$

Integrating (3.9) from t_6 to t and using the substitution $x = y_3(w)$ from (3.9) we get

$$\lim_{t\to\infty} \int_{y_3(t_6)}^{y_3(t)} \frac{dx}{x^{\alpha_1\alpha_2\alpha_3}} \leqslant -\int_{t_6}^{\infty} p_3(v)\Big[\int_{t_5}^{h_1(v)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{\alpha_3} dv. \tag{3.10}$$

We know that $y_3(t)$ is a decreasing function and $y_3(t) > 0$. Thus $\lim_{t\to\infty} y_3(t) = K_1 \geqslant 0$ and in view of (A2) we obtain $\lim_{t\to\infty} \int_{y_3(t_6)}^{y_3(t)} \frac{dx}{x^{\alpha_1\alpha_2\alpha_3}} = K_2$, where K_2 is a finite real number. This fact contradicts the assumption (A3).

I.2.b Let $y_2(t) < 0, t \geqslant t_4 \geqslant t_3$. In regard of (c) there exists a $t_5 \geqslant t_4$ such that $y_2(h_2(t)) < 0$, for $t \geqslant t_5$. The assumptions (d), (b) and the first equation of (1.1) imply that $z(t)$ is a decreasing function for $t \geqslant t_5$. On the interval $[t_5, \infty)$ hold:

- $y_1(t) > 0$;
- $z(t)$ is a decreasing function and $z(t) > 0$;
- $y_2(t)$ is an increasing function and $y_2(t) < 0$;
- $y_3(t)$ is a decreasing function and $y_3(t) > 0$.

Therefore exist $\lim_{t\to\infty} y_3(t) = A \geqslant 0,\ \lim_{t\to\infty} y_2(t) = B \leqslant 0$ and $\lim_{t\to\infty} z(t) = C \geqslant 0$. We shall show that $A = 0, B = 0$ and $C = 0$.

(i) Let $A > 0$. Then $y_3(t) \geqslant A$ for $t \geqslant T_0 \geqslant t_5$. In view of (c) and raising to the power of α_2 we have $(y_3(h_3(t)))^{\alpha_2} \geqslant A^{\alpha_2}$ for $t \geqslant T_1 \geqslant T_0$. Integrating the second equation of (1.1) from T_1 to t and using the last inequality we get

$$y_2(t) - y_2(T_1) \geqslant A^{\alpha_2} \int_{T_1}^{t} p_2(s)\, ds, \quad t \geqslant T_1. \tag{3.11}$$

(3.11) and (b) imply that $\lim_{t\to\infty} y_2(t) = \infty$. Therefore $y_2(t) > 0$ for $t \geqslant T_2 \geqslant T_1$, which contradicts $y_2(t) < 0$ for $t \geqslant t_5$. Then $\lim_{t\to\infty} y_3(t) = 0$.

(ii) Assume that $B < 0$. Then $y_2(t) \leqslant B$ for $t \geqslant T_0 \geqslant t_5$ and in regard of (c) we have $y_2(h_2(t)) \leqslant B$ for $t \geqslant T_1 \geqslant T_0$. Hence $| y_2(h_2(t)) |= -y_2(h_2(t)) \geqslant K_1, K_1 = -B, t \geqslant T_1$. Raising this inequality to the power of α_1, multiplying by $-p_1(t)$ and using the first equation of (1.1) we obtain

$$z'(t) \leqslant -K_1^{\alpha_1} p_1(t), \quad t \geqslant T_1.$$

Integrating the last inequality from T_1 to t and in view of (b) we get $\lim_{t\to\infty} z(t) = -\infty$. Therefore $z(t) < 0$ for $t \geqslant T_2 \geqslant T_1$ which is a contradiction with positivity of $z(t)$ for $t \geqslant t_5$.

(iii) Let $C > 0$. Then $z(t) \geqslant C$ for $t \geqslant T_0 \geqslant t_5$. Taking into account the definition of $z(t)$ we are led to $y_1(t) \geqslant z(t) \geqslant C$ for $t \geqslant T_0$. In view of (c) we have $y_1(h_1(t)) \geqslant C$ for $t \geqslant T_1 \geqslant T_0$ and the third equation of (1.1) implies

$$y_3'(t) \leqslant -C^{\alpha_3} p_3(t), \quad t \geqslant T_1.$$

Integrating the last inequality from T_1 to t and multiplying by (-1) we obtain

$$y_3(T_1) \geqslant y_3(T_1) - y_3(t) \geqslant C^{\alpha_3} \int_{T_1}^{t} p_3(s)\, ds, \quad t \geqslant T_1.$$

Hence for $t \to \infty$ we get

$$y_3(T_1) \geqslant C^{\alpha_3} \int_{T_1}^{\infty} p_3(s)\, ds. \tag{3.12}$$

In view of (c) there exists a $T_2 \geqslant T_1$ such that $h_3(t) \geqslant T_1$ for $t \geqslant T_2$. Then (3.12) holds for $h_3(t)$, $t \geqslant T_2$, too:

$$y_3(h_3(t)) \geqslant C^{\alpha_3} \int_{h_3(t)}^{\infty} p_3(s)\, ds, \quad t \geqslant T_2 \geqslant T_1.$$

Using the second equation of (1.1) we have

$$y_2'(t) \geqslant C^{\alpha_2\,\alpha_3} p_2(t)\Big(\int\limits_{h_3(t)}^{\infty} p_3(s)\,ds\Big)^{\alpha_2}, \quad t \geqslant T_2. \tag{3.13}$$

Integrating (3.13) from T_2 to t and in regard of (A4) we obtain $\lim\limits_{t\to\infty} y_2(t) = \infty$. Hence $y_2(t) > 0$ pre $t \geqslant T_3 \geqslant T_2$ which is a contradiction with $y_2(t) < 0$ for $t \geqslant t_5$. Therefore $\lim\limits_{t\to\infty} z(t) = 0$ and from Lemma 2.3 we obtain that $\lim\limits_{t\to\infty} y_1(t) = 0$.

II. Let $y_1(t) \in N^-$, $t \geqslant t_0$. Then $z(t) < 0, t \geqslant t_0$. Using the assumptions (c), (d) and (b), the third equation of (1.1) implies that $y_3(t)$ is a decreasing function for $t \geqslant t_1 \geqslant t_0$.

II.1 Assume that $y_3(t) < 0$, $t \geqslant t_2 \geqslant t_1$. Then we can proceed the same way as in the case I.1 to get $\lim\limits_{t\to\infty} z(t) = -\infty$ which is contrary to Lemma 2.2.

II.2 Let $y_3(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. In view of (c) there exists a $t_3 \geqslant t_2$ such that $y_3(h_3(t)) > 0$ for $t \geqslant t_3$. The assumptions (d),(b) and the second equation of (1.1) imply that $y_2(t)$ is an increasing function for $t \geqslant t_3$.

II.2.a Let $y_2(t) > 0$ for $t \geqslant t_4 \geqslant t_3$. In regard of (c) and monotonicity of $y_2(t)$ holds: $y_2(h_2(t)) \geqslant y_2(t_4)$ for $t \geqslant t_5 \geqslant t_4$. Raising this inequality to the power of α_1, multiplying by $p_1(t)$ and using the first equation of (1.1) we get $z'(t) \geqslant M^{\alpha_1} p_1(t)$ where $M = y_2(t_4)$, $t \geqslant t_5$. Integrating this inequality from t_5 to t we obtain

$$z(t) - z(t_5) \geqslant M^{\alpha_1} \int\limits_{t_5}^{t} p_1(s)ds, \quad t \geqslant t_5.$$

Hence $\lim\limits_{t\to\infty} z(t) = \infty$ which is a contradiction with Lemma 2.2.

II.2.b Let $y_2(t) < 0$ for $t \geqslant t_4 \geqslant t_3$. In view of assumptions (c), (d), (b) and first equation of (1.1) we get that $z(t)$ is a decreasing function for $t \geqslant t_5 \geqslant t_4$. Therefore $\lim\limits_{t\to\infty} z(t) = A < 0$ which contradicts the Lemma 2.2. □

Theorem 3.2. *Let $\sigma = -1$ and assume that (A1) and (A4) hold. Moreover, let*

(A5) $\alpha_1\,\alpha_2\,\alpha_3 = 1$;

(A6)

$$\int^{\infty} p_3(t)\Big[\int\limits_0^{h_1(t)} p_1(u)\Big(\int\limits_0^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du\Big]^{(1-\epsilon)\alpha_3} dt = \infty, \quad 0 < \epsilon < 1.$$

Then every proper solution $y \in W$ of (1.1) is either oscillatory or $y_i(t)$, i=1,2,3 tend monotonically to zero as $t \to \infty$.

Proof. Assume that $y(t) \in W$ is a nonoscillatory solution of (1.1) and $y_1(t) > 0$ for $t \geqslant t_0$. We can proceed exactly as in the proof of Theorem 3.1. We shall discuss only the possibility I.2.a. The proofs of cases I.1, I.2.b and II. are the same.

I. Let $y_1(t) \in N^+, t \geqslant t_0$. Then $z(t) > 0, t \geqslant t_0$ and the third equation of (1.1) implies that $y_3(t)$ is a decreasing function for $t \geqslant t_1 \geqslant t_0$.

I.2 Assume that $y_3(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. The assumptions (c), (d), (b) and the second equation of (1.1) imply that $y_2(t)$ is an increasing function for $t \geqslant t_3$.

I.2.a Let $y_2(t) > 0$ for $t \geqslant t_4 \geqslant t_3$. Then we can proceed the same way as for the case I.2.a of Theorem 3.1 to get (3.7):

$$\Big(y_1(h_1(t))\Big)^{\alpha_3} \geqslant \Big(y_3(h_3(h_2(h_1(t))))\Big)^{\alpha_1\alpha_2\alpha_3}\Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{\alpha_3}$$

for $t \geqslant t_6$. In view of monotonicity of $y_3(t)$, assumptions (A1), (A5) and raising to the power of $1-\epsilon$ we are led to

$$(y_1(h_1(t)))^{(1-\epsilon)\,\alpha_3} \geqslant (y_3(t))^{1-\epsilon}\Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{(1-\epsilon)\,\alpha_3} \tag{3.14}$$

for $t \geqslant t_6$. The property $y_2(t) > 0, t \geqslant t_4$ and the first equation of (1.1) imply that $z(t)$ is an increasing function for all sufficiently large t. From the proof of Theorem 3.1 we know that $h_1(t) \geqslant t_5$ for $t \geqslant t_6$. Therefore $z(h_1(t)) \geqslant z(t_5)$ for $t \geqslant t_6$ and from $y_1(t) \geqslant z(t)$, $t \geqslant t_0$ we get $y_1(h_1(t)) \geqslant z(t_5)$, $t \geqslant t_6$. Hence

$$1 \geqslant \frac{K_1}{(y_1(h_1(t)))^{\alpha_3}}, \quad K_1 = (z(t_5))^{\alpha_3} > 0,\, t \geqslant t_6.$$

Raising to the power of ϵ and multiplying by $(y_1(h_1(t)))^{\alpha_3}$ may be the last inequality rewritten as

$$(y_1(h_1(t)))^{(1-\epsilon)\,\alpha_3} \leqslant K_2\,(y_1(h_1(t)))^{\alpha_3}, \text{ kde } K_2 = K_1^{-\epsilon},\, t \geqslant t_6.$$

Combining this inequality and (3.14), multiplying by $-p_3(t)$ and using the third equation of (1.1) we obtain

$$K_2\,(y_3(t))^{\epsilon-1}\,y_3'(t) \leqslant -p_3(t)\Big[\int_{t_5}^{h_1(t)} p_1(u)\Big(\int_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1} du\Big]^{(1-\epsilon)\,\alpha_3},\, t \geqslant t_6. \tag{3.15}$$

Integrating (3.15) from t_6 to t we have

$$\frac{K_2}{\epsilon}\Big[(y_3(t))^{\epsilon}-(y_3(t_6))^{\epsilon}\Big] \leqslant -\int\limits_{t_6}^{t} p_3(x)\Big[\int\limits_{t_5}^{h_1(x)} p_1(u)\Big(\int\limits_{t_4}^{h_2(u)} p_2(s)\,ds\Big)^{\alpha_1}du\Big]^{(1-\epsilon)\,\alpha_3}dx$$

for $t \geqslant t_6$.

The last inequality and the assumption (A6) imply that $\lim\limits_{t\to\infty}(y_3(t))^{\epsilon}=-\infty$. But $(y_3(t))^{\epsilon}$ is a decreasing function and $(y_3(t))^{\epsilon}\geqslant 0$. Therefore $\lim\limits_{t\to\infty}(y_3(t))^{\epsilon}= A\geqslant 0$ and this is a contradiction with $\lim\limits_{t\to\infty}(y_3(t))^{\epsilon}=-\infty$. □

Theorem 3.3. *Assume that $\sigma=1$ and the assumptions (A3), (A4) of Theorem 3.1 are fulfilled. Then every proper solution $y\in W$ of (1.1) is either oscillatory or $|\,y_i(t)\,|$, $i=1,2,3$ tend monotonically to infinity as $t\to\infty$ or $y_i(t)$, i=1,2,3 tend monotonically to zero as $t\to\infty$.*

Proof. Let $y(t)\in W$ be a nonoscillatory solution of (1.1). According to Lemma 2.1 there exists a $t_0\geqslant 0$ such that $z(t)$, $y_2(t)$, $y_3(t)$ are monotone functions of constant sign on the interval $[t_0,\infty)$. Without loss of generality we may assume that $y_1(t)>0$ for $t\geqslant t_0$. Then either $y_1(t)\in N^+$ or $y_1(t)\in N^-$ for $t\geqslant t_0$.

I. Let $y_1(t)\in N^+, t\geqslant t_0$. Therefore $z(t)>0$ for $t\geqslant t_0$. Using the assumptions (c), (d) and (b), the system (1.1) implies that the following four cases may occur:

I.1	$y_1(t)>0$	$y_2(t)$ is increasing and $y_2(t)>0$	$y_3(t)$ is increasing and $y_3(t)>0$	$z(t)$ is increasing and $z(t)>0$
I.2	$y_1(t)>0$	$y_2(t)$ is increasing and $y_2(t)<0$	$y_3(t)$ is increasing and $y_3(t)>0$	$z(t)$ is decreasing and $z(t)>0$
I.3	$y_1(t)>0$	$y_2(t)$ is decreasing and $y_2(t)>0$	$y_3(t)$ is increasing and $y_3(t)<0$	$z(t)$ is increasing and $z(t)>0$
I.4	$y_1(t)>0$	$y_2(t)$ is decreasing and $y_2(t)<0$	$y_3(t)$ is increasing and $y_3(t)<0$	$z(t)$ is decreasing and $z(t)>0$

I.1 In view of (c) and monotonicity of $y_3(t)$ we get $y_3(h_3(t))\geqslant y_3(t_5)$ for $t\geqslant t_6\geqslant t_5$. Raising this inequality to the power of α_2, multiplying by $p_2(t)$ and using the second equation of (1.1) we have:

$$y_2'(t)\geqslant L_1^{\alpha_2}p_2(t),\ L_1=y_3(t_5),\ t\geqslant t_6.$$

Integrating the last equation from t_6 to t we obtain

$$y_2(t)\geqslant y_2(t)-y_2(t_6)\geqslant L_1^{\alpha_2}\int\limits_{t_6}^{t}p_2(s)ds,\quad t\geqslant t_6. \tag{3.16}$$

Hence $\lim\limits_{t\to\infty} y_2(t)=\infty$, i.e. $\lim\limits_{t\to\infty} |y_2(t)|=\infty$.

In regard of (c) and monotonicity of $y_2(t)$ we are led to $y_2(h_2(t)) \geqslant y_2(t_5)$, $t \geqslant t_6 \geqslant t_5$. Raising this inequality to the power of α_1, multiplying by $p_1(t)$ and using the first equation of (1.1) we get:

$$z'(t) \geqslant L_2^{\alpha_1} p_1(t),\ t \geqslant t_6,\ L_2 = y_2(t_5).$$

Integrating the last inequality from t_6 to t and using $y_1(t) \geqslant z(t)$ for $t \geqslant t_0$ we have:

$$y_1(t) \geqslant L_2^{\alpha_1} \int\limits_{t_6}^{t} p_1(s)ds, \qquad t \geqslant t_6.$$

Therefore $\lim\limits_{t\to\infty} y_1(t)=\infty$ and $\lim\limits_{t\to\infty} |y_1(t)|=\infty$.

In view of (c) there exists a $t_7 \geqslant t_6$ such that $h_2(t) \geqslant t_6$ for $t \geqslant t_7$. Then (3.16) holds for $h_2(t), t \geqslant t_7$, too:

$$y_2(h_2(t)) \geqslant L_1^{\alpha_2} \int\limits_{t_6}^{h_2(t)} p_2(s)ds, \quad t \geqslant t_7.$$

Hence we have

$$z'(t) = p_1(t)\Big(y_2(h_2(t))\Big)^{\alpha_1} \geqslant L_3\, p_1(t)\Big(\int\limits_{t_6}^{h_2(t)} p_2(s)ds\Big)^{\alpha_1},\ \ L_3 = L_1^{\alpha_1\,\alpha_2},\ t \geqslant t_7.$$

Integrating this inequality from t_7 to t and taking into account $y_1(t) \geqslant z(t)$ we get

$$y_1(t) \geqslant L_3 \int\limits_{t_7}^{t} p_1(u)\Big(\int\limits_{t_6}^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du,\ \ t \geqslant t_7. \tag{3.17}$$

In regard of (c) the last inequality holds for $h_1(t)$, $t \geqslant t_8 \geqslant t_7$, too:

$$y_1(h_1(t)) \geqslant L_3 \int\limits_{t_7}^{h_1(t)} p_1(u)\Big(\int\limits_{t_6}^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du,\ \ t \geqslant t_8.$$

Hence using the third equation of (1.1) we obtain

$$y_3'(t) \geqslant L_4\, p_3(t)\Big(\int\limits_{t_7}^{h_1(t)} p_1(u)\Big(\int\limits_{t_6}^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du\Big)^{\alpha_3},\ \ L_4 = L_3^{\alpha_3},\ t \geqslant t_8. \tag{3.18}$$

Integrating (3.18) from t_8 to t we get

$$y_3(t) \geqslant L_4 \int\limits_{t_8}^{t} p_3(v)\Big(\int\limits_{t_7}^{h_1(v)} p_1(u)\Big(\int\limits_{t_6}^{h_2(u)} p_2(s)ds\Big)^{\alpha_1} du\Big)^{\alpha_3} dv, \ \ t \geqslant t_8.$$

In view of (A3) the last inequality implies $\lim\limits_{t\to\infty} y_3(t) = \infty$. Then $\lim\limits_{t\to\infty} |y_3(t)| = \infty$.

I.2 We can proceed the same way as for the case I.1 to get (3.16):

$$y_2(t) \geqslant y_2(t) - y_2(t_6) \geqslant L_1^{\alpha_2} \int\limits_{t_6}^{t} p_2(s)ds, \quad t \geqslant t_6.$$

Therefore $\lim\limits_{t\to\infty} y_2(t) = \infty$, i.e. $y_2(t) > 0$ for $t \geqslant t_7 \geqslant t_6$. But this is a contradiction with $y_2(t) < 0$ for $t \geqslant t_5$.

I.3 Using (c), monotonicity of $z(t)$ and $y_1(t) \geqslant z(t)$ we have: $y_1(h_1(t)) \geqslant L_5$, $L_5 = z(t_5)$, $t \geqslant t_6 \geqslant t_5$. Then the third equation of (1.1) may be rewritten as $y_3'(t) \geqslant L_5^{\alpha_3} p_3(t)$, $t \geqslant t_6$. Integrating this inequality from t_6 to t we obtain:

$$-y_3(t_6) \geqslant y_3(t) - y_3(t_6) \geqslant L_5^{\alpha_3} \int\limits_{t_6}^{t} p_3(s)ds, \quad t \geqslant t_6.$$

Hence for $t \to \infty$ we see that

$$-y_3(t_6) \geqslant L_5^{\alpha_3} \int\limits_{t_6}^{\infty} p_3(s)ds.$$

In regard of (c) the last inequality holds for $h_3(t)$, $t \geqslant t_7 \geqslant t_6$, too:

$$-y_3(h_3(t)) = |y_3(h_3(t))| \geqslant L_6 \int\limits_{h_3(t)}^{\infty} p_3(s)ds, \ \ L_6 = L_5^{\alpha_3}, \ t \geqslant t_7.$$

Hence

$$y_2'(t) = -p_2(t)|y_3(h_3(t))|^{\alpha_2} \leqslant -L_6^{\alpha_2} p_2(t)\Big(\int\limits_{h_3(t)}^{\infty} p_3(s)ds\Big)^{\alpha_2}, \quad t \geqslant t_7,$$

and integrating from t_7 to t we are led to

$$y_2(t) - y_2(t_7) \leqslant -L_6^{\alpha_2} \int\limits_{t_7}^{t} p_2(u)\Big(\int\limits_{h_3(u)}^{\infty} p_3(s)ds\Big)^{\alpha_2} du, \quad t \geqslant t_7.$$

Therefore in view of (A4) we get $\lim_{t\to\infty} y_2(t) = -\infty$. It means that $y_2(t) < 0$ for $t \geqslant t_8 \geqslant t_7$ which is contrary to $y_2(t) > 0$ for $t \geqslant t_5$.

I.4 In regard of (c) and monotonicity of $y_2(t)$ we have $|y_2(h_2(t))| \geqslant L_7$, $L_7 = (-y_2(t_5))$, $t \geqslant t_6 \geqslant t_5$. Hence $z'(t) = -p_1(t)|y_2(h_2(t))|^{\alpha_1} \leqslant -L_7^{\alpha_1} p_1(t)$, $t \geqslant t_6$ and integrating from t_6 to t we obtain

$$z(t) - z(t_6) \leqslant -L_7^{\alpha_1} \int_{t_6}^{t} p_1(s)ds, \quad t \geqslant t_6.$$

Using (b) the last inequality imply that $\lim_{t\to\infty} z(t) = -\infty$. Therefore $z(t) < 0$ for $t \geqslant t_7 \geqslant t_6$ which is a contradiction with $z(t) > 0$ for $t \geqslant t_5$.

II. Let $y_1(t) \in N^-$. Hence $z(t) < 0$ for $t \geqslant t_0$ and the third equation of (1.1) implies that $y_3(t)$ is an increasing function for $t \geqslant t_1$.

II.1 Assume that $y_3(t) > 0$, $t \geqslant t_2 \geqslant t_1$. Then $y_3(h_3(t)) > 0$ for $t \geqslant t_3 \geqslant t_2$ and from the second equation of (1.1) we get that $y_2(t)$ is an increasing function for $t \geqslant t_3$.

II.1.a Let $y_2(t) > 0$ for $t \geqslant t_4$. In view of (c) and monotonicity of $y_2(t)$ we have $(y_2(h_2(t)))^{\alpha_1} \geqslant (y_2(t_4))^{\alpha_1}$ for $t \geqslant t_5 \geqslant t_4$. Integrating the first equation of (1.1) from t_5 to t and using the last inequality we are led to

$$z(t) - z(t_5) \geqslant (y_2(t_4))^{\alpha_1} \int_{t_5}^{t} p_1(s)ds, \quad t \geqslant t_5.$$

Hence in view of (b) we get $\lim_{t\to\infty} z(t) = \infty$ which contradicts Lemma 2.2.

II.1.b Let $y_2(t) < 0, t \geqslant t_4$. Taking into account assumptions (b), (c), (d) the first equation of (1.1) implies that $z(t)$ is a decreasing function for $t \geqslant t_5$. It means that $\lim_{t\to\infty} z(t) = A < 0$ which is contrary to Lemma 2.2.

II.2 Assume that $y_3(t) < 0, t \geqslant t_2 \geqslant t_1$. From the second equation of (1.1) we get that $y_2(t)$ is a decreasing function for $t \geqslant t_3$.

Function $y_3(t)$ is increasing. Therefore exists $\lim_{t\to\infty} y_3(t) = B \leqslant 0$. We shall show that $B = 0$.

Let $B < 0$. Then $y_3(h_3(t)) \leqslant B < 0$ for $t \geqslant t_4 \geqslant t_3$. Hence $|y_3(h_3(t))| \geqslant C$, $C = -B$ and

$$y_2'(t) = -p_2(t)|y_3(h_3(t))|^{\alpha_2} \leqslant -C^{\alpha_2} p_2(t), \quad t \geqslant t_4.$$

Integrating the last inequality from t_4 to t and using (b) we obtain $\lim_{t\to\infty} y_2(t) = -\infty$, i.e. $y_2(t) < 0, t \geqslant t_5 \geqslant t_4$. In regard of assumptions (b), (c) and (d) the first equation of (1.1) implies that $z(t)$ is a decreasing function for $t \geqslant t_6$. Therefore $\lim_{t\to\infty} z(t) = D < 0$ which is a contradiction with Lemma 2.2. Then $\lim_{t\to\infty} y_3(t) = 0$.

II.2.a Let $y_2(t) < 0$, $t \geqslant t_4$. From the first equation of (1.1) we have that $z(t)$ is a decreasing function. Therefore $\lim_{t\to\infty} z(t) = E < 0$ which contradicts Lemma 2.2.

II.2.b If $y_2(t) > 0$, $t \geqslant t_4 \geqslant t_3$, then exists $\lim_{t\to\infty} y_2(t) = F \geqslant 0$. We shall show that $F = 0$.

Assume that $F > 0$. Then $y_2(h_2(t)) > F$, $t \geqslant t_5 \geqslant t_4$ and hence

$$z'(t) = p_1(t)(y_2(h_2(t)))^{\alpha_1} > F^{\alpha_1} p_1(t), \qquad t \geqslant t_5.$$

Integrating the last inequality from t_5 to t and using (b) we obtain $\lim_{t\to\infty} z(t) = \infty$. Therefore $z(t) > 0$ for $t \geqslant t_6 \geqslant t_5$ which is a contradiction with $z(t) < 0$. Then $\lim_{t\to\infty} y_2(t) = 0$.

Because $y_2(t) > 0$, the first equation of (1.1) implies that $z(t)$ is an increasing function such that $z(t) < 0$. In regard of Lemma 2.2 we obtain $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} y_1(t) = 0$. □

References

[1] Győri, I., Ladas, G., Oscillation Theory Of Delay Differential Equations, Clarendon Press, Oxford, 1991.

[2] Marušiak, P., Oscillatory properties of functional differential systems of neutral type, *Czechoslovak Math. J.,* **43** (118), (1993), 649–662.

[3] Mihalíková, B., A note on the asymptotic properties of systems of neutral differential equations, *Stud. Univ. Zilina, Math. Phys. Ser.,* Vol. 13, (2001), 133–139.

[4] Mihalíková, B., Oscillations of neutral differential systems, *Discuss. Math. Differential Incl.,* Vol. 19, (1999), 5–15.

[5] Mihalíková, B., Some properties of neutral differential systems equations, *Bollettino U. M. I.,* Vol. 8 (5-B), (2002), 279–287.

[6] Mihály, T., On the oscillatory and asymptotic properties of solutions of systems of neutral differential equations, *Nonlinear Analysis: Theory, Methods & Applications* (to appear)

[7] Shevelo, V. N., Varech, N. V., Gritsai, A. G., Oscillatory properties of solutions of systems of differential equations with deviating arguments, *preprint no. 85.10,* Institute of Mathematics of the Ukrainian Academy of Sciences Russian, (1985).

[8] Špániková, E., Asymptotic properties of solutions of nonlinear differential systems with deviating argument, Doctoral Thesis, University of Žilina, Žilina, Slovakia, (1990).

[9] Špániková, E., Oscillatory properties of solutions of three-dimensional differential systems of neutral type, *Czechoslovak Math. J.,* **50** (125), (2000), 879–887.

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