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A generalization of the Barabási-Albert random tree*

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Abstract

In this paper a random graph evolution rule is defined which can be considered as a generalization of the Barabási-Albert random tree. The evolution is a combination of the preferential attachment method and the interactions of 2 vertices. Our model is similar to the 3-interactions model studied in [2]. We describe the asymptotic behaviour of the degrees and the weights of the vertices.

Keywords: Random graph, preferential attachment, scale-free, power law

MSC: 05C80, 60G42

1. Introduction

Several real life networks are scale-free (see [4, 7]). A random graph is called scale-free, if it has a power law degree distribution, that is $P(d) \sim d^{-\gamma}$ as $d \to \infty$, where $P(d)$ is the probability that a vertex is of degree d. The well-known Barabási-Albert preferential attachment model produces a scale-free sequence of random graphs.

The Barabási-Albert model

The preferential attachment model was suggested by Barabási and Albert in [4]. See also the paper of Yule [17] for trees. The graph evolution rule given in [4] is

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the following. The starting point is a graph with a small number of vertices. At every time step a new vertex is added with m edges that link the new vertex to m different vertices already present in the graph. The preferential attachment means that the probability $p(i)$ that the new vertex will be connected to vertex i depends on the degree of that vertex, so that $p(i) = k_i / \sum_j k_j$, where k_j denotes the degree of vertex j. According to [5], the model is not defined precisely by this definition. A precise definition of the model and a rigorous proof of the scale-free property was given in [5] (see also [7, 16]). The simplest case of the model is the Barabási-Albert random tree, when $m = 1$.

In [6] a generalization of the Barabási-Albert model was introduced. In [6] besides the preferential attachment method, uniform choice of vertices are allowed, moreover, new connections can be grown between old vertices. For the recent results in the preferential attachment model see [16, 13, 10].

The 3-interactions model

In [2] the following graph evolution was introduced. We start with a single triangle. This graph contains 3 vertices and 3 edges. Each of these objects has initial weight 1. The evolution of the graph is based on the interactions of three vertices. At each step we consider three vertices and we draw all non-existing edges between them. So we obtain a triangle. The weight of this triangle and the weights all of its edges and vertices are increased by 1.

At a fixed time the evolution is the following. Independently of the past, with probability p, a new vertex is born which interacts with 2 old vertices. That is they form a triangle. The two old vertices can be chosen in two different ways. With probability r we choose an edge from the existing edges according to their weights. The two vertices of that edge will interact with the new vertex. On the other hand, with probability $1-r$, we choose 2 from the existing vertices uniformly. They will interact with the new vertex. Independently of the past, with probability $1 - p$, we do not add a new vertex, but three of the old vertices interact. To select the three old vertices we have two options. With probability q we choose one out of the existing triangles according to their weights. The vertices of the triangle chosen will interact. On the other hand, with probability $1-q$, we choose from the existing vertices uniformly (that is all three vertices have the same chance).

The power law degree distribution in that model was proved in [2] and [3]. The model and the results were extended to N-interactions model in [8] and [9], if $N \geq 4$.

The goal of this paper

In this paper a random graph evolution mechanism is defined. The evolution of the graph is a combination of the preferential attachment and the interaction of 2 vertices. A vertex in our graph is characterized by its degree and its weight. The weight of a given vertex is the number of the interactions of the vertex. The asymptotic behaviour of the graph is studied. Scale-free properties both for the degrees and the weights are proved. The proofs are based on discrete time martingale theory.

Our model is a special case of the N-interactions model of [8] and [9]. However, our result can not be obtained as a particular case of the general results of [8] and [9] because the basic equation for the 2-interactions model is not a special case of the basic equation for the N-interactions model with $N \geq 3$. In this paper we follow the method elaborated by Backhausz and Móri in [2, 3]. We do not present detailed proofs because they are similar to the ones in [2, 3, 8, 9] .

2. The 2-interactions random graph model and the main results

In this paper we study the following version of the Barabási-Albert random tree.

At time $n = 0$ we start with two connected vertices. The initial weights of the

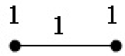

Figure 1: $n = 0$, the initial state

two vertices and the initial weight of the edge are equal to one. The weights of the non-existing edges and vertices are always considered to be 0. The evolution of the graph is based on the interactions of 2 vertices. At each step $n = 1, 2, \ldots$ we consider 2 vertices and if they are not connected, then we draw the edge between them. The weights of the two vertices and the weight of the edge connecting them are increased by 1.

The evolution of the graph is the following. On the one hand, with probability p, we add a new vertex, that will interact with 1 old vertex. On the other hand, with probability $(1 - p)$, we do not add any new vertex, but 2 old vertices interact.

(a) If we add a new vertex, then we choose 1 old vertex which will interact with the new one. To choose the old vertex we have two possibilities. With probability r we choose a vertex from the existing vertices according to the weights of the vertices. That is a vertex k with weight w_k has chance $w_k/(\sum_l w_l)$. On the other hand, with probability $1 - r$, we choose from the existing vertices uniformly, that is any vertex has the same chance.

(b) At the step when we do not add a new vertex, then 2 old vertices interact. To select the 2 old vertices we have two options. With probability q we choose one edge from the existing edges according to their weights. That is the probability that we choose an edge is proportional to its weight. Then the two vertices of that edge will interact. On the other hand, with probability $1 - q$, we choose two out of the existing vertices uniformly. That is all two vertices have the same chance.

Figure 2 shows an example for the graph evolution. At the initial step $n = 0$ we have an edge and two vertices. At step $n = 1$ we add a new vertex with initial weight 1, choose an old vertex and connect them using a new edge. The initial

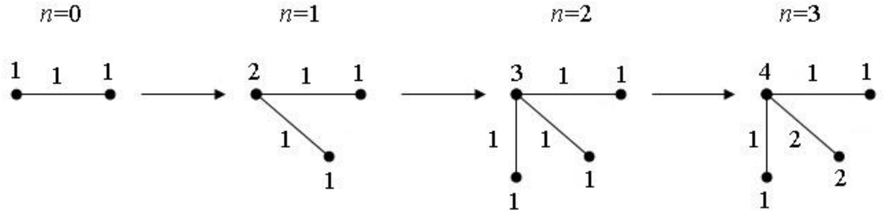

Figure 2: An example for the graph evolution

weight of the new edge is 1 and we increase the weight of the old vertex by 1. Step 2 is similar to step 1. However, at step $n = 3$, we do not add a new vertex, but 2 old vertices interact. We choose two out of the existing vertices, then increase the weights of the vertices and the weight of the edge connecting them by 1. So we can see that the weight of a given vertex is the number of the interactions of the vertex.

Our results are confined to the 2-interactions model. To describe the main results we need the following notation. Throughout the paper $0 < p < 1$, $0 \le r \le 1$, $0 \le q \le 1$ are fixed numbers. Let $X(n, d, w)$ denote the number of vertices of weight w and degree d after the nth step. Let V_n denote the number of vertices after the nth step.

Each vertex has initial weight 1 and initial degree 1. When a vertex takes part in an interaction, then its weight is increased by 1 and its degree may increase by 0 or 1. So $X(n, d, w)$ can be positive only for $1 \le w \le n + 1$ and $1 \le d \le w$.

Let

$$\alpha_1 = (1-p)\,q, \quad \alpha_2 = pr/2, \quad \alpha = \alpha_1 + \alpha_2, \\ \beta = (1-r) + 2\,(1-p)\,(1-q)/p. \tag{2.1}$$

The following theorem describes the limiting behaviour of the relative frequency of vertices with a fixed weight and a fixed degree.

Theorem 2.1. *Let $0 < p < 1$, $q > 0$. Assume that at least one of the following three conditions are satisfied: $r > 0$ or $r < 1$ or $q < 1$. Then for any fixed w and d with $1 \le w$ and $1 \le d \le w$ we have*

$$X\,(n, d, w)/V_n \to x_{d,w} \tag{2.2}$$

almost surely as $n \to \infty$, where $x_{d,w}$ are fixed positive numbers. Furthermore, the numbers $x_{d,w}$ satisfy the following recurrence relation

$$x_{1,1} = 1/(\alpha + \beta + 1) > 0, \qquad x_{d,1} = 0, \ \text{for } d \ne 1,$$

$$x_{d,w} = \frac{1}{\alpha w + \beta + 1}\left[\alpha_1\,(w-1)\,x_{d,w-1} + (\alpha_2\,(w-1) + \beta)\,x_{d-1,w-1}\right], \tag{2.3}$$

for $w \ge 2$, $1 \le d \le w$. If $1 \le d \le w$ is not satisfied, then $x_{d,w} = 0$.

The following lemma states that the numbers $x_{d,w}$, $d = 1, \dots, w$, $w = 1, 2, \dots$, form a (proper) two-dimensional discrete probability distribution. Moreover, its marginal distributions will be the limiting distributions of the weights and the degrees, respectively.

Lemma 2.2. *Let $p > 0$ and define $x_w = x_{1,w} + x_{2,w} + \dots + x_{w,w}$ for $w = 1, 2, \dots$. Then x_w, $w = 1, 2, \dots$, are positive numbers satisfying the following recurrence relation*

$$x_1 = \frac{1}{\alpha + \beta + 1}, \quad x_w = \frac{\alpha (w-1) + \beta}{\alpha w + \beta + 1} x_{w-1}, \quad \text{if} \quad w > 1. \tag{2.4}$$

x_w, $w = 1, 2, \dots$, is a discrete probability distribution. Moreover, $x_{d,w}$, $d = 1, \dots, w$, $w = 1, 2, \dots$, is a two-dimensional discrete probability distribution.

Next theorem shows the scale-free property of the weights of the vertices.

Theorem 2.3. *Let $X(n, w)$ denote the number of vertices of weight w after n steps. Assume that the conditions of Theorem 2.1 are satisfied. Then for all $w = 1, 2, \dots$ we have*

$$X(n, w)/V_n \to x_w \tag{2.5}$$

almost surely, as $n \to \infty$, where x_w, $w = 1, 2, \dots$, are positive numbers satisfying the recurrence relation (2.4). Moreover,

$$x_w \sim C w^{-\left(1+\frac{1}{\alpha}\right)} \quad \text{as} \quad w \to \infty \tag{2.6}$$

with $C = \Gamma\left(1 + \frac{\beta+1}{\alpha}\right) / \left(\alpha \Gamma\left(1 + \frac{\beta}{\alpha}\right)\right)$.

Our main result is the scale-free property of the degrees.

Theorem 2.4. *Assume that the conditions $0 < p < 1$, $q > 0$, and $r > 0$ are satisfied. Let us denote by $U(n, d)$ the number of vertices of degree d after n steps, that is $U(n, d) = \sum_{w: d \le w \le n+1} X(n, d, w)$. Then, for any $d \ge 1$ we have*

$$\frac{U(n, d)}{V_n} \to u_d \tag{2.7}$$

a.s. as $n \to \infty$, where $u_d = \sum_w x_{d,w}$, $d = 1, 2, \dots$, are positive numbers. Furthermore,

$$u_d \sim \frac{\Gamma\left(1 + \frac{\beta+1}{\alpha}\right)}{\alpha_2 \Gamma\left(1 + \frac{\beta}{\alpha}\right)} \left(\frac{\alpha}{\alpha_2}\right)^{-\left(1+\frac{1}{\alpha}\right)} d^{-\left(1+\frac{1}{\alpha}\right)} \quad \text{as} \quad d \to \infty. \tag{2.8}$$

3. Proofs and auxiliary results

The following lemma contains the basic equation of the paper. Let $\mathcal{F}_{n-1}$ denote the σ-algebra of observable events after $(n-1)$ steps. We compute the conditional expectation of $X(n, d, w)$ with respect to $\mathcal{F}_{n-1}$ for $w \ge 1$.

Lemma 3.1.

$$E(X(n,d,w)|\mathcal{F}_{n-1}) = X(n-1,d,w)\left(1-\frac{\alpha w}{n}-\beta\frac{p}{V_{n-1}}\right)+ \tag{3.1}$$
$$+X(n-1,d,w-1)(1-p)\left[q\frac{w-1}{n}+(1-q)\frac{d}{\binom{V_{n-1}}{2}}\right]+$$
$$+X(n-1,d-1,w-1)\left[p\left[r\frac{w-1}{2n}+(1-r)\frac{1}{V_{n-1}}\right]+(1-p)(1-q)\frac{V_{n-1}-d}{\binom{V_{n-1}}{2}}\right]+$$
$$+p\delta_{d,1}\delta_{w,1}.$$

Here $\delta_{a,b}$ denotes the Dirac-delta.

Proof. The probability that an old vertex of weight w takes part in the interaction at step n is

$$p\left(r\frac{w}{2n}+(1-r)\frac{1}{V_{n-1}}\right)+(1-p)\left(q\frac{w}{n}+(1-q)\frac{V_{n-1}-1}{\binom{V_{n-1}}{2}}\right)=\frac{w}{n}\alpha+\frac{p}{V_{n-1}}\beta,$$

where α and β are defined by (2.1). So the terms at the right hand side of (3.1) correspond to the following cases. The first term covers the case when neither the degree nor the weight of a vertex change. Its probability is $1-\left(\frac{\alpha w}{n}+\beta\frac{p}{V_{n-1}}\right)$. The second term covers the case when the degree does not change but the weight is increased by 1, while the third term correspond to the case when both the degree and the weight are increased by 1. A new vertex always takes part in the interaction. At each step, with probability p, a new vertex with weight 1 and with degree 1 is born. This explains term $p\delta_{d,1}\delta_{w,1}$ in (3.1). □

We shall need the following results on discrete time martingales. Let $\{Z_n,\mathcal{F}_n\}$ be a submartingale. Its Doob-Meyer decomposition is $Z_n = M_n + A_n$, where $\{M_n,\mathcal{F}_n\}$ is a martingale and $\{A_n,\mathcal{F}_n\}$ is an increasing predictable process. Here, up to an additive constant,

$$A_n = \mathbb{E}Z_1+\sum_{i=2}^{n}(\mathbb{E}(Z_i|\mathcal{F}_{i-1})-Z_{i-1}).$$

We see that $\{M_n^2,\mathcal{F}_n\}$ is again a submartingale. Let

$$M_n^2 = Y_n + B_n$$

be the Doob-Meyer decomposition of M_n^2. Here, up to an additive constant,

$$B_n=\sum_{i=2}^{n}\mathbb{D}^2(Z_i|\mathcal{F}_{i-1})=\sum_{i=2}^{n}\mathbb{E}\left\{(Z_i-\mathbb{E}(Z_i|\mathcal{F}_{i-1}))^2|\mathcal{F}_{i-1}\right\}.$$

Proposition 3.1 (Propositions VII-2-3 and VII-2-4 of [12]). *Let $M_1 = 0$. On the set $\{B_\infty < \infty\}$ the martingale M_n almost surely converges to a finite limit. Moreover, $M_n = \mathrm{o}(B_n^{1/2} \log B_n)$ almost surely on the set $\{B_n \to \infty\}$.*

A consequence of the above proposition is the following.

Proposition 3.2 (Proposition 2.3 of [1]). *Let $\{Z_n, \mathcal{F}_n\}$ be a square integrable non-negative submartingale. If $B_n^{1/2} \log B_n = \mathrm{O}(A_n)$, then $Z_n \sim A_n$ as $n \to \infty$, almost surely on the set $\{A_n \to \infty\}$.*

Proof of Theorem 2.1. Applying the Marcinkiewicz strong law of large numbers to the number of vertices, we obtain

$$V_n = pn + \mathrm{o}\left(n^{1/2+\varepsilon}\right) \tag{3.2}$$

almost surely, for any $\varepsilon > 0$. Let

$$c(n,w) = \prod_{i=1}^{n} \left(1 - \frac{\alpha w}{i} - \frac{\beta p}{V_{i-1}}\right)^{-1}.$$

Then (3.2) and Taylor's expansion imply that

$$c(n,w) \sim a_w n^{\alpha w + \beta} \tag{3.3}$$

almost surely as $n \to \infty$, where a_w is a positive random variable.

Let $Z(n,d,w) = c(n,w)X(n,d,w)$. Then, by (3.1), $(Z(n,d,w), \mathcal{F}_n)$ is a non-negative submartingale. We shall apply the Doob-Meyer decompositions $Z_n = M_n + A_n$ and $M^2 = Y_n + B_n$. Then

$$\begin{aligned} A(n,d,w) = {} & EZ(1,d,w) + \\ & + \sum_{i=2}^{n} c(i,w) X(i-1,d,w-1)(1-p)\left(q\frac{w-1}{i} + (1-q)\frac{d}{\binom{V_{i-1}}{2}}\right) + \\ & + \sum_{i=2}^{n} c(i,w) X(i-1,d-1,w-1) \times \\ & \times \left[p\left(r\frac{w-1}{2i} + (1-r)\frac{1}{V_{i-1}}\right) + (1-p)(1-q)\frac{V_{i-1}-d}{\binom{V_{i-1}}{2}}\right] + \\ & + \sum_{i=2}^{n} c(i,w) p \delta_{d,1}\delta_{w,1}. \end{aligned} \tag{3.4}$$

Moreover

$$B(n,d,w) = \sum_{i=2}^{n} \mathbb{D}^2\left(Z(i,d,w) \,|\mathcal{F}_{i-1}\right) \leq$$

$$\leq \sum_{i=2}^{n} c(i,w)^2\, \mathbb{E}\{(X(i,d,w) - X(i-1,d,w))^2 \,|\mathcal{F}_{i-1}\} \leq$$
$$\leq 4\sum_{i=2}^{n} c(i,w)^2 = \mathrm{O}\left(n^{2(\alpha w+\beta)+1}\right). \tag{3.5}$$

We use induction on w. Let $w = 1$. We see that a vertex of weight 1 took part in an interaction only when it was born. Therefore its degree must be equal to 1. By (3.4),

$$A(n,1,1) \sim p\sum_{i=2}^{n} c(i,1) \sim p\sum_{i=2}^{n} a_1 i^{\alpha+\beta} \sim pa_1 \frac{n^{\alpha+\beta+1}}{\alpha+\beta+1} \tag{3.6}$$

a.s. as $n \to \infty$. By (3.5), $B(n,1,1) = \mathrm{O}\left(n^{2(\alpha+\beta)+1}\right)$ and therefore

$$(B(n,1,1))^{\frac{1}{2}} \log B(n,1,1) = \mathrm{O}(A(n,1,1)).$$

It follows from Proposition 3.2 that

$$Z(n,1,1) \sim A(n,1,1) \ \text{ a.s. on the event } \ \{A(n,1,1) \to \infty\} \text{ as } n \to \infty. \tag{3.7}$$

As, by (3.6), $A(n,1,1) \to \infty$ a.s., therefore using the asymptotic behaviour of V_n and $c(n,w)$, relation (3.7) implies

$$\frac{X(n,1,1)}{V_n} = \frac{Z(n,1,1)}{c(n,1)\,V_n} \sim \frac{A(n,1,1)}{c(n,1)\,V_n} \sim \frac{pa_1 \frac{n^{\alpha+\beta+1}}{\alpha+\beta+1}}{a_1 n^{\alpha+\beta} pn} = \frac{1}{\alpha+\beta+1} = x_{1,1} > 0$$

almost surely. So (2.2) is valid for $w = 1$.

Suppose that the statement is true for all weights less than w and for all possible degrees. It implies that $X(n,d,w-1) \sim x_{d,w-1} np$.

Then by (3.2), (3.3) and using the induction hypothesis, we have for any $w > 1$

$$A(n,d,w) \sim \sum_{i=2}^{n} \left[c(i,w)\, x_{d,w-1} pi\,(1-p)\, q \frac{w-1}{i} + \right.$$
$$\left. + c(i,w)\, x_{d-1,w-1} pi \left(pr\frac{(w-1)}{2i} + \frac{p(1-r)}{pi} + \frac{2(1-p)(1-q)}{pi} \right) \right] \sim$$
$$\sim \sum_{i=2}^{n} a_w i^{\alpha w+\beta} \left[x_{d,w-1} p(1-p)\, q(w-1) + \right.$$
$$\left. + x_{d-1,w-1} \left(\frac{1}{2} p^2 r(w-1) + p(1-r) + 2(1-p)(1-q) \right) \right] \sim$$
$$\sim pa_w \frac{n^{\alpha w+\beta+1}}{\alpha w+\beta+1} \left[(1-p)\, q(w-1)\, x_{d,w-1} + \right.$$
$$\left. + \left(\frac{1}{2} pr(w-1) + \ (1-r) + \frac{2(1-p)(1-q)}{p} \right) x_{d-1,w-1} \right]. \tag{3.8}$$

In the above computation we deleted all terms having asymptotically smaller degree than the largest one.

Formula (3.8) implies $A(n,d,w) \sim pa_w n^{\alpha w+\beta+1} x_{d,w} \to \infty$, because $x_{d,w} > 0$, where, by (3.8),

$$x_{d,w} = \frac{1}{\alpha w+\beta+1}\left[\alpha_1 (w-1) x_{d,w-1} + (\alpha_2 (w-1)+\beta) x_{d-1,w-1}\right],$$

with α_1, α_2, α and β defined by (2.1). Therefore $(B(n,d,w))^{\frac{1}{2}} \log B(n,d,w) = \mathrm{O}(A(n,d,w))$. So, using Proposition 3.2, we have $Z(n,d,w) \sim A(n,d,w)$. Therefore

$$\frac{X(n,d,w)}{V_n} = \frac{Z(n,d,w)}{c(n,w)V_n} \sim \frac{A(n,d,w)}{c(n,w)V_n} \sim \frac{pa_w n^{\alpha w+\beta+1} x_{d,w}}{a_w n^{\alpha w+\beta} pn} = x_{d,w} \tag{3.9}$$

a.s. as $n \to \infty$. □

Proof of Lemma 2.2. If $\alpha = 0$, then the statement is obvious. Now assume $\alpha \neq 0$. As $x_{d,w}$ is defined as $x_{d,w} = 0$ for $d \notin \{1,2,\ldots,w\}$, therefore $x_w = \sum_d x_{d,w}$. From the recurrence relation (2.3) we obtain

$$\begin{aligned} x_w &= \sum_{d=1}^{w} x_{d,w} = \sum_d x_{d,w} = \\ &= \frac{1}{\alpha w+\beta+1}\left[\alpha_1 (w-1)\sum_d x_{d,w-1} + (\alpha_2 (w-1)+\beta)\sum_d x_{d-1,w-1}\right] = \\ &= \frac{\alpha(w-1)+\beta}{\alpha w+\beta+1} x_{w-1}. \end{aligned}$$

Using this recursive formula for x_w, we obtain

$$\begin{aligned} x_w &= x_1 \prod_{j=2}^{w} \frac{\alpha(j-1)+\beta}{\alpha j+\beta+1} = \frac{1}{\alpha w+\beta+1} \prod_{j=1}^{w-1} \frac{\frac{\beta}{\alpha}+j}{\frac{\beta+1}{\alpha}+j} = \\ &= \frac{\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{\alpha\Gamma\left(1+\frac{\beta}{\alpha}\right)} \frac{\Gamma\left(w+\frac{\beta}{\alpha}\right)}{\Gamma\left(w+\frac{\beta+1}{\alpha}+1\right)}. \end{aligned} \tag{3.10}$$

By [15], we have the following formula:

$$\sum_{k=0}^{n} \frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{a-b+1}\left[\frac{\Gamma(n+a+1)}{\Gamma(n+b)} - \frac{\Gamma(a)}{\Gamma(b-1)}\right].$$

Therefore, by some calculation, we obtain $\sum_{w=1}^{n} x_w \to 1$ as $n \to \infty$. So $\sum_{w=1}^{\infty} x_w = 1$. As $\sum_d x_{d,w} = x_w$, so $\sum_{w=1}^{\infty}\sum_{d=1}^{w} x_{d,w} = 1$ and therefore $x_{d,w}$, $d = 1,2,\ldots,w$, $w = 1,2,\ldots$, is a (proper) two-dimensional discrete probability distribution. □

Proof of Theorem 2.3. As

$$X(n,w) = X(n,1,w) + X(n,2,w) + \cdots + X(n,w,w),$$

Theorem 2.1 and Lemma 2.2 imply (2.5). Using (3.10), the Stirling formula gives (2.6). □

The following representation of the joint distribution of degrees and weights is useful to prove scale-free property for degrees. Let W be a random variable with distribution $\mathbb{P}(W = w) = x_w$, $w = 1, 2, \ldots$. Let $\xi_1 \equiv 1$ and $\xi_2, \xi_3, \ldots$ be independent random variables being independent of W, too. For $w \geq 2$ let ξ_w have the following distribution:

$$\mathbb{P}(\xi_w = 0) = \frac{\alpha_1 (w-1)}{\alpha (w-1) + \beta}, \quad \mathbb{P}(\xi_w = 1) = \frac{\alpha_2 (w-1) + \beta}{\alpha (w-1) + \beta}.$$

Let $S_w = \xi_1 + \xi_2 + \cdots + \xi_w$.

Lemma 3.2. $\mathbb{P}(S_W = d, W = w) = x_{d,w}$ *for all* $w = 1, 2, \ldots,\ d = 1, 2, \ldots, w$.

Proof. It is easy to see that the sequence $\mathbb{P}(S_W = d, W = w)$ satisfies the same recursion (2.3) as $x_{d,w}$. □

To obtain scale-free property for degrees, we need the following local limit theorem. Let $X_1, X_2, \ldots$ be independent, integer valued random variables. Let $p_{j,m} = \mathbb{P}(X_j = m)$ be the distribution, while $p_{j,m_j} = \max_m p_{j,m}$ be the maximal value of the distribution. Let $S_n = \sum_{i=1}^n X_i$ be the partial sum, $P_n(N) = \mathbb{P}(S_n = N)$ be its distribution, $M_n = \sum_{i=1}^n \mathbb{E}X_i$ be the expectation, and $B_n = \sum_{i=1}^n \mathbb{E}(X_i - \mathbb{E}X_i)^2$ be the variance of S_n.

Proposition 3.3 (Theorem 5 and its consequence in Section VII, 2 of [14]). *Assume that the greatest common divisor of the values*

$$\left\{ m \,:\, \frac{1}{\log n} \sum_{j=1}^n p_{j,m_j} p_{j,m+m_j} \to \infty \right\}$$

is equal to 1. Moreover,

$$\liminf \frac{B_n}{n} > 0, \quad \limsup \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i - \mathbb{E}X_i|^3 < \infty.$$

Then

$$\sup_N \left| \sqrt{B_n} P_n(N) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(N - M_n)^2}{2B_n} \right) \right| = \mathrm{O}\left(\frac{1}{\sqrt{n}} \right).$$

If we apply Proposition 3.3 to the random variables ξ_k in Lemma 3.2, then we obtain the following result which will play an important role in the proof our main theorem.

Proposition 3.4. *Suppose that $\alpha_1 > 0$ and $\alpha_2 > 0$. Then*

$$x_{d,w} = x_w \frac{\alpha}{\sqrt{2\pi\alpha_1\alpha_2 w}} \left[\exp\left(-\frac{(d-\mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right) + \mathrm{O}\left(w^{-\frac{1}{2}}\right)\right] \quad \textit{as} \quad w \to \infty, \tag{3.11}$$

where the error term $\mathrm{O}\left(w^{-\frac{1}{2}}\right)$ *does not depend on* d.

Proof. We follow the method of *Theorem 4.2* in [3]. Let $w > 1$. Then we have

$$\mathbb{E}\xi_w = \frac{\alpha_2 (w-1) + \beta}{\alpha (w-1) + \beta} = \frac{\alpha_2}{\alpha} + \frac{\alpha_1 \beta}{\alpha (\alpha (w-1) + \beta)},$$

hence

$$\mathbb{E}S_w = \mathbb{E}\xi_1 + \cdots + \mathbb{E}\xi_w = w\frac{\alpha_2}{\alpha} + \mathrm{O}(\log w) \tag{3.12}$$

as $w \to \infty$. By simple computation, we obtain

$$\mathbb{D}^2\xi_w = \frac{\alpha_1\alpha_2}{\alpha^2} + \mathrm{O}\left(\frac{1}{w}\right), \quad \mathbb{D}^2 S_w = \frac{\alpha_1\alpha_2}{\alpha^2} w + \mathrm{O}(\log w) \tag{3.13}$$

as $w \to \infty$.

Now, we apply Proposition 3.3 for S_w. The conditions of that proposition are satisfied, therefore we have

$$\sup_{d\in\mathbb{Z}} \left| \mathbb{D}S_w \, \mathbb{P}(S_w = d) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d-\mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right)\right| = \mathrm{O}\left(\frac{1}{\sqrt{w}}\right). \tag{3.14}$$

Using (3.13) and (3.14), we obtain $\left|\mathbb{D}S_w - \frac{\sqrt{\alpha_1\alpha_2 w}}{\alpha}\right| \mathbb{P}(S_w = d) = \mathrm{O}\left(w^{-\frac{1}{2}}\right)$. Therefore (3.14) implies that

$$\sup_{d\in\mathbb{Z}} \left| \frac{\sqrt{\alpha_1\alpha_2 w}}{\alpha} \mathbb{P}(S_w = d) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d-\mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right)\right| = \mathrm{O}\left(\frac{1}{\sqrt{w}}\right). \tag{3.15}$$

By the independence of W and ξ_i, we see that $x_{d,w} = \mathbb{P}(S_W = d, W = w) = \mathbb{P}(S_w = d)\, x_w$. So the result follows from (3.15). □

The well-known Hoeffding's inequality is the following.

Proposition 3.5 (Theorem 2 of [11])**.** *Let $X_1, X_2, \ldots, X_n$ be independent random variables, $a_i \le X_i \le b_i$ ($i = 1, 2, \ldots, n$). Let $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$, $\mu = \mathbb{E}\bar{X}$. Then for any $t > 0$*

$$\mathbb{P}(\bar{X} - \mu \ge t) \le \exp\left(\frac{-2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

Proof of Theorem 2.4. Theorem 2.1 and Lemma 2.2 will imply (2.7). Hoeffding's inequality, Lemma 3.2 and Proposition 3.4 will imply (2.8).

By Theorem 2.1 and Lemma 3.2, $\frac{X(n,d,w)}{V_n}$ converges almost surely to the distribution $x_{d,w} = \mathbb{P}(S_W = d, W = w)$. But the cardinalities of terms in the sum $\sum_{w:d\le w\le n+1} X(n,d,w)$ are not bounded when $n\to\infty$. However, using that $x_{d,w}$, $d = 1,2,\ldots,w$, $w = 1,2,\ldots$ is a proper two-dimensional discrete distribution, the convergence of the marginal distributions is a consequence of the convergence of the two-dimensional distributions. So we obtain (2.7).

To obtain (2.8), we can apply the method of *Theorem 4.3* in [3]. Let

$$f = \frac{\alpha}{\alpha_2} d\,, \quad H = H_d = \left\{ w : f - f^{\frac{1}{2}+\varepsilon} \le w \le f + f^{\frac{1}{2}+\varepsilon} \right\},$$

$$H^- = H_d^- = \left\{ w : w < f - f^{\frac{1}{2}+\varepsilon} \right\}, \quad H^+ = H_d^+ = \left\{ w : w > f + f^{\frac{1}{2}+\varepsilon} \right\}$$

with some fixed $0 < \varepsilon < 1/6$.

Using (3.12) and Proposition 3.5, we obtain for $w \in H^-$

$$\mathbb{P}(S_w = d) \le \mathbb{P}(S_w \ge d) \le \mathbb{P}\left(S_w - \mathbb{E}S_w \ge d - \frac{\alpha_2}{\alpha} w - \mathrm{O}(\log w)\right) \le$$
$$\le \exp\left\{ -\frac{2}{w}\left(d - \frac{\alpha_2}{\alpha} w - \mathrm{O}(\log w)\right)^2 \right\} = \exp\left\{ -2\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{(f - w - \mathrm{O}(\log w))^2}{w} \right\}.$$

Now $w \in H^-$ implies that

$$(f - w - \mathrm{O}(\log w))^2 = (f-w)^2 - 2(f-w)\,\mathrm{O}(\log w) + (\mathrm{O}(\log w))^2 \ge$$
$$\ge f^{1+2\varepsilon} - \mathrm{O}(f \log f)\,.$$

Therefore in the case when $w \in H^-$ we obtain

$$\mathbb{P}(S_w = d) \le \exp\left\{ -2\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{f^{1+2\varepsilon} - \mathrm{O}(f\log f)}{f} \right\} =$$
$$= \exp\left\{ -2\left(\frac{\alpha_2}{\alpha}\right)^2 f^{2\varepsilon} + \mathrm{O}(\log f) \right\}.$$

This implies that

$$\mathbb{P}\left(S_W = d, W \in H^-\right) = \sum_{w\in H^-} \mathbb{P}(S_w = d, W = w) \le \sum_{w\in H^-} \mathbb{P}(S_w = d) \le$$
$$\le f \exp\left\{ -2\left(\frac{\alpha_2}{\alpha}\right)^2 f^{2\varepsilon} + \mathrm{O}(\log f) \right\} = \mathrm{o}\left(f^{-\left(1+\frac{1}{\alpha}\right)}\right). \tag{3.16}$$

In the case when $w \in H^+$, by Hoeffding's inequality, we have

$$\mathbb{P}(S_w = d) \le \mathbb{P}(S_w \le d) \le \mathbb{P}\left(S_w - \mathbb{E}S_w \le d - \frac{\alpha_2}{\alpha} w\right) \le$$

$$\leq \exp\left\{-\frac{2}{w}\left(d-\frac{\alpha_2}{\alpha}w\right)^2\right\}=\exp\left\{-2\left(\frac{\alpha_2}{\alpha}\right)^2\frac{(f-w)^2}{w}\right\}.$$

Because $w \in H^+$ and $\frac{1}{2}+\varepsilon<1$, we obtain $2(w-f)\geq f^{\frac{1}{2}+\varepsilon}+w-f\geq f^{\frac{1}{2}+\varepsilon}+(w-f)^{\frac{1}{2}+\varepsilon}\geq w^{\frac{1}{2}+\varepsilon}$. So

$$\mathbb{P}(S_w=d)\leq\exp\left\{-2\left(\frac{\alpha_2}{\alpha}\right)^2\frac{w^{1+2\varepsilon}}{4w}\right\}=\exp\left\{-\frac{1}{2}\left(\frac{\alpha_2}{\alpha}\right)^2 w^{2\varepsilon}\right\}.$$

Therefore

$$\mathbb{P}\left(S_W=d,W\in H^+\right)\leq\sum_{\{w\,:\,f<w\}}\exp\left\{-\frac{1}{2}\left(\frac{\alpha_2}{\alpha}\right)^2 w^{2\varepsilon}\right\}=\mathrm{o}\left(f^{-\left(1+\frac{1}{\alpha}\right)}\right). \qquad (3.17)$$

Now turn to the case of $w\in H=H_d$. Consider the set

$$B=\{(d,w)\,:\,w\geq 1,d\geq 1,w\in H_d\}.$$

It is easy to see that

$$\text{if}\quad d\to\infty\quad\text{and}\quad(d,w)\in B,\quad\text{then}\quad\frac{w}{d}\to 1.$$

As $w\in H$, so we have $w=f+\mathrm{O}\left(f^{\frac{1}{2}+\varepsilon}\right)$. Then (with $\varepsilon_1>0$ arbitrarily small)

$$-\frac{(d-\mathbb{E}S_w)^2}{2\mathbb{D}^2S_w}=-\frac{\left(d-w\frac{\alpha_2}{\alpha}-\mathrm{O}(\log w)\right)^2}{2\frac{\alpha_1\alpha_2}{\alpha^2}w+\mathrm{O}(\log w)}=-\frac{\alpha_2}{\alpha_1}\frac{(f-w-\mathrm{O}(\log w))^2}{2w+\mathrm{O}(\log w)}=$$

$$=-\frac{\alpha_2}{\alpha_1}\frac{(f-w)^2+\mathrm{O}\left(f^{\frac{1}{2}+\varepsilon+\varepsilon_1}\right)}{2w+\mathrm{O}(\log w)}=-\frac{\alpha_2}{\alpha_1}\frac{(f-w)^2}{2f}+\mathrm{O}\left(f^{-\frac{1}{2}+3\varepsilon}\right) \qquad (3.18)$$

as $d\to\infty$. Here the error term does not depend on w. By (3.11), (2.6) and (3.18), we obtain

$$x_{d,w}\sim$$

$$\sim Cw^{-\left(1+\frac{1}{\alpha}\right)}\frac{\alpha}{\sqrt{2\pi\alpha_1\alpha_2 w}}\left[\exp\left\{-\frac{\alpha_2}{\alpha_1}\frac{(f-w)^2}{2f}+\mathrm{O}\left(f^{-\frac{1}{2}+3\varepsilon}\right)\right\}+\mathrm{O}\left(w^{-\frac{1}{2}}\right)\right]\sim$$

$$\sim Cf^{-\left(1+\frac{1}{\alpha}\right)}\frac{\alpha}{\alpha_2}\frac{1}{\sqrt{2\pi\frac{\alpha_1}{\alpha_2}f}}\exp\left\{-\frac{(f-w)^2}{2\frac{\alpha_1}{\alpha_2}f}\right\}$$

as $d\to\infty$ and $w\in H$, where $C=\Gamma\left(1+\frac{\beta+1}{\alpha}\right)/\left(\alpha\Gamma\left(1+\frac{\beta}{\alpha}\right)\right)$. Therefore

$$\sum_{w\in H}x_{d,w}\sim\sum_{f-f^{\frac{1}{2}+\varepsilon}<w<f+f^{\frac{1}{2}+\varepsilon}}Cf^{-\left(1+\frac{1}{\alpha}\right)}\frac{\alpha}{\alpha_2}\frac{1}{\sqrt{2\pi\frac{\alpha_1}{\alpha_2}f}}\exp\left\{-\frac{(f-w)^2}{2\frac{\alpha_1}{\alpha_2}f}\right\}=$$

$$= Cf^{-\left(1+\frac{1}{\alpha}\right)}\frac{\alpha}{\alpha_2}\sum_{-f^{\frac{1}{2}+\varepsilon}<k<f^{\frac{1}{2}+\varepsilon}}\frac{1}{\sqrt{2\pi\frac{\alpha_1}{\alpha_2}f}}\exp\left\{-\frac{k^2}{2\frac{\alpha_1}{\alpha_2}f}\right\} =$$

$$= A\sum_{-f^{\varepsilon}<\frac{k}{\sqrt{f}}<f^{\varepsilon}}\frac{1}{\sqrt{f}}\frac{1}{\sqrt{2\pi\frac{\alpha_1}{\alpha_2}}}\exp\left\{-\frac{\left(\frac{k}{\sqrt{f}}\right)^2}{2\frac{\alpha_1}{\alpha_2}}\right\} \to$$

$$\to A\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\frac{\alpha_1}{\alpha_2}}}\exp\left\{-\frac{x^2}{2\frac{\alpha_1}{\alpha_2}}\right\}dx = A,$$

where

$$A = \frac{\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{\alpha_2\Gamma\left(1+\frac{\beta}{\alpha}\right)}\left(\frac{\alpha d}{\alpha_2}\right)^{-\left(1+\frac{1}{\alpha}\right)}.$$

So we obtain

$$\mathbb{P}\left(S_W = d, W \in H\right) \sim \frac{\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{\alpha_2\Gamma\left(1+\frac{\beta}{\alpha}\right)}\left(\frac{\alpha d}{\alpha_2}\right)^{-\left(1+\frac{1}{\alpha}\right)} \tag{3.19}$$

as $d \to \infty$. Finally, by (3.16), (3.17) and (3.19), we obtain

$$u_d \sim \frac{\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{\alpha_2\Gamma\left(1+\frac{\beta}{\alpha}\right)}\left(\frac{\alpha}{\alpha_2}d\right)^{-\left(1+\frac{1}{\alpha}\right)}$$

as $d \to \infty$. □

References

[1] Backhausz, Á., *Analysis of random graphs with methods of martingale theory.* PhD thesis, Eötvös Loránd University, Budapest, 2012.

[2] Backhausz, Á., Móri, T. F., A random graph model based on 3-interactions. *Ann. Univ. Sci. Budapest. Sect. Comput.* Vol. 36 (2012), 41–52.

[3] Backhausz, Á., Móri, T. F., Weights and degrees in a random graph model based on 3-interactions. *Acta Math. Hungar.* Vol. 143 (2014), no. 1, 23–43.

[4] Barabási, A.L., Albert, R., Emergence of scaling in random networks. *Science*, Vol. 286 (1999), 509–512.

[5] Bollobás, B., Riordan, O., Spencer, J., Tusnády, G., The degree sequence of a scale-free random graph process. *Random Structures Algorithms*, Vol. 18 (2001), 279–290.

[6] Cooper, C., Frieze, A., A general model of web graphs. *Random Structures Algorithms*, Vol. 22 (2003), 311–335.

[7] Durrett, R., *Random graph dynamics.* Cambridge University Press, Cambridge UK, 2007.

[8] Fazekas, I., Porvázsnyik, B., Scale-free property for degrees and weights in a preferential attachment random graph model. *Journal of Probability and Statistics,* Vol. 2013 (2013), Article ID 707960.

[9] Fazekas, I., Porvázsnyik, B., Scale-free property for degrees and weights in an N-interaction random graph model. *arXiv:1309.4258v1* [math.PR] 17 Sep. 2013.

[10] Grechnikov, E., An estimate for the number of edges between vertices of given degrees in random graphs in the Bollobás-Riordan model. *Mosc. J. Comb. Number Theory*, Vol. 1 (2011), no. 2, 40–73.

[11] Hoeffding, W., Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, Vol. 58 (1963), 13–30.

[12] Neveu, J., *Discrete-parameter martingales.* North-Holland, Amsterdam, 1975.

[13] Ostroumova, L., Ryabchenko, A., Samosvat, E., Generalized preferential attachment: tunable power-law degree distribution and clustering coefficient. *arXiv*:1205.3015v1 [math.CO] 14 May 2012.

[14] Petrov, V. V., *Sums of Independent Random Variables.* Akademie-Verlag, Berlin, 1975.

[15] Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I., *Integrals and series.* Gordon & Breach Science Publishers, New York, 1986.

[16] van der Hofstad, R., *Random Graphs and Complex Networks.* Eindhoven University of Technology, The Netherlands, rhofstad@win.tue.nl, 2013.

[17] Yule, G. U., A Mathematical Theory of Evolution, Based on the Conclusions of Dr. J. C. Willis, F.R.S. *Phil. Transact. Royal Society London, Ser. B*, Vol. 213 (1925), 21–87.