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# Power spectrum estimation of spherical random fields based on covariances*

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**Abstract**

A Gaussian isotropic stochastic field on a 2D-sphere is characterized by either its covariance function or its angular spectrum. The object of this paper is the estimation of the spectrum in two steps. First we estimate the covariance function, secondly we approximate the series expansion of the covariance function with respect of Legendre polynomials. Simulations show that this method is fast and precise.

*Keywords:* Angular correlation, angular spectrum, isotropic fields on sphere, estimation of correlation

*MSC:* 60G60, 62M30

## 1. Introduction

There are several physical phenomena which can be described with the help of a spherical random processes. A typical example of random data measured on the surface of a sphere is the cosmic microwave background radiation (CMB). Similar random fields arise in medical imaging, in analysis of gravitational and geomagnetic data etc.. These fields are characterized by a series expansion with respect to the spherical harmonics. Under assumption of Gaussianity both the covariance function and the angular power spectrum describe completely the probability structure of

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an isotropic stochastic field. The estimated spectrum can be used to check the underlying physical theory, while the possible non-Gaussianity can be investigated by estimating the higher order angular spectra.

## 1.1. Notations

Let $\mathbb{S}_2$ denote the surface of the unit sphere in $\mathbb{R}^3$, and $X(L)$ be a random field on $\mathbb{S}_2$, where the location $L = (\vartheta, \varphi)$, and $\vartheta \in [0, \pi]$ is the co-latitude, while $\varphi \in [0, 2\pi]$ is the longitude. If the spatial process $X(L)$ is mean square continuous, then it has a series expansion in terms of spherical harmonics $Y_\ell^m$. Spherical harmonics are defied by the Legendre polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell, \quad x \in [-1, 1],$$

$(\ell = 0, 1, 2, \dots)$ and the associated Legendre functions

$$P_\ell^m = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x),$$

of degree $\ell$ and order $m$, where $\ell = 0, 1, 2, \dots,$ and $m = -\ell, \dots, \ell$. Now the complex valued spherical harmonics of degree $\ell$ and order $m$ ($\ell = 0, 1, 2, \dots,$ and $m = -\ell, \dots, \ell$) are given by

$$Y_\ell^m(\vartheta, \varphi) = \lambda_\ell^m(cos\vartheta) e^{im\varphi},$$

where

$$\lambda_\ell^m(x) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(x), \quad \text{if } m \geq 0,$$

and

$$\lambda_\ell^m(x) = (-1)^m \lambda_\ell^{|m|}(x), \quad \text{if } m < 0,$$

that implies

$$Y_\ell^{-m}(\vartheta, \varphi) = (-1)^m \overline{Y_\ell^m}(\vartheta, \varphi).$$

Using these notations the spherical harmonics expansion of the random field $X(L) \in L^2(\mathbb{S}_2)$ is

$$X(L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(L),$$

where the coefficients

$$Z_\ell^m, \quad \ell = 0, 1, \dots, \quad m = -\ell, \dots, \ell$$

are complex valued centered random variables, while putting $EZ_0^0 = \mu$ implies that $EX(L) = \mu$ and the coefficients are given by

$$Z_\ell^m = \int_{\mathbb{S}^2} X(L) \overline{Y_\ell^m}(L) dL, \tag{1.1}$$

and

$$Z_\ell^m = (-1)^m \overline{Z_\ell^{-m}}.$$

## 2. Spectrum

**Definition.** The random field $X(L)$ is called strongly isotropic if all finite dimensional distributions of $\{X(L), L \in \mathbb{S}_2\}$ are invariant under the rotation $g$ for every $g \in SO(3)$, where $SO(3)$ denotes the special orthogonal group of rotations defined on $\mathbb{S}_2$.

If the spatial process $X$ is strongly isotropic, then

$$E(Z_{\ell_1}^{m_1} \overline{Z_{\ell_2}^{m_2}}) = f_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$

for $\ell_1, \ell_2 \in \mathbb{N}$, and $m_i = -\ell_i, \ldots, \ell_i$, where $\delta_{\ell k} = 1$ if $\ell = k$ and zero otherwise, while

$$E(Z_0^0 \overline{Z_\ell^m}) = \left(f_0 + E(Z_0^0)^2\right) \delta_{0\ell} \delta_{0m}, \quad f_0 = Var\left(Z_0^0\right).$$

$f_\ell = Var\left(Z_\ell^m\right)$, $\ell = 0, 1, 2, \ldots$, are nonnegative real numbers, and $(f_\ell, \ell \in \mathbb{N}_0)$ is called the *angular power spectrum* of the random field $X$. Note $E(X) = \mu$, hence $C_2(L_1, L_2) = E(X(L_1) - \mu)(X(L_2) - \mu)$ is the covariance function of the isotropic field $X(L)$. Due to the isotropy the covariance $C_2(L_1, L_2)$ depends on the angular distance $\gamma$ of the locations $L_1$ and $L_2$ only (where $\cos\gamma = L_1 \cdot L_2$). That means

$$C_2(L_1, L_2) = C_2(g_{L_2 L_1} L_1, N) =: C(\cos\gamma),$$

where $g_{L_2 L_1}$ is the rotation which takes $L_2$ into the north pole $N$ and $L_1$ into the plane $xOz$.

It is straightforward (see [5]) that

$$C(\cos\gamma) = \sum_0^\infty f_\ell \frac{2\ell + 1}{4\pi} P_\ell(\cos\gamma). \tag{2.1}$$

For the practical computation of the spectrum $f_k$ the orthogonality of the Legendre polynomials can be used: with $t = \cos(\gamma)$ from (2.1) follows

$$\int_{-1}^{1} C(t) P_\ell(t) dt = f_\ell \frac{2\ell + 1}{4\pi} \int_{-1}^{1} [P_\ell(t)]^2 dt = f_\ell \cdot \frac{1}{2\pi},$$

that is

$$f_\ell = 2\pi \int_{-1}^{1} C(t) P_\ell(t) dt, \tag{2.2}$$

for $\ell = 0, 1, 2, \ldots$.

**Example** (Lapalce-Beltrami model on $\mathbb{S}_2$)**.** Consider the homogeneous isotropic field $X$ on $\mathbb{R}^3$ according to the equation

$$\left(\triangle - c^2\right) X = \partial W,$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, denotes the Laplace operator on $\mathbb{R}^3$. Its spectrum, see ([6]), is

$$S(\lambda) = \frac{2}{(2\pi)^2} \frac{\lambda^2}{\left(\lambda^2 + c^2\right)^2}, \quad \lambda^2 = \left\|(\lambda_1, \lambda_2, \lambda_3)\right\|^2,$$

with covariance of Matérn Class

$$\mathcal{C}(r) = \frac{1}{(2\pi)^{3/2}} \frac{(cr)^{1/2} K_{1/2}(cr)}{2c},$$

where $K_{1/2}$ is the modified Bessel (Hankel) function, see [1].
Now according to the Lapalce-Beltrami operator, which is the restriction of $\triangle$ onto the unit sphere $\mathbb{S}_2$,

$$\triangle_B = \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2},$$

we consider the stochastic model

$$\left(\triangle_B - c^2\right) X_B = \partial W_B,$$

on sphere. The covariance function $\mathcal{C}_0$ of $X_B$ is the restriction of the covariance function $\mathcal{C}$ of $X$ on sphere and $\mathcal{C}_0(\cos\gamma) = \mathcal{C}(2\sin(\gamma/2))$, i.e.

$$\mathcal{C}_0(\cos\gamma) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\sin(\gamma/2)}{2c}} K_{1/2}(2c\sin(\gamma/2)).$$

We apply the Poisson formula when $\Phi(d\lambda) = S(\lambda)\, d\lambda$, and we obtain the spectrum for $X_B$

$$\begin{aligned} f_\ell &= 2\pi^2 \int_0^\infty J_{\ell+1/2}^2(\lambda) \frac{1}{\lambda} \frac{2}{(2\pi)^2} \frac{\lambda^2}{\left(\lambda^2 + c^2\right)^2} d\lambda \\ &= \int_0^\infty J_{\ell+1/2}^2(\lambda) \frac{\lambda}{\left(\lambda^2 + c^2\right)^2} d\lambda. \end{aligned}$$

# 3. HEALPix

The most widely used pixelisation of the sphere for sampling and analyzing CMB data is the HEALPix (Hierarchical, Equal Area and isoLatitude Pixelization), see

[2]. Actually the CMB data are given on the surface of a unit ball at the discrete points defined by HEALPix. Here in the base resolution partitioning the surface of the sphere is divided into 12 quadrilateral pixels of same area, and in each further resolution the pixels are subdivided into 4 equal area pixels. Denoting by $N_{side}$ the resolution parameter, the total number of pixels equals $12N_{side}^2$, and the pixel centers are located on $4N_{side}-1$ isolatitude rings. Unfortunately, the pixelisation is not rotational invariant, the pixel centers can be rotated into each other in the case of some rotations around the north-south axes only.

# 4. Computational results

Let us suppose that we are given an observation of an isotropic field on the sphere, more precisely for each HEALPix pixel $L$ we have a value $X(L)$. The estimator of the spectrum of the field can be based either on (1.1) or on (2.1). It means that we can approximate the integral (1.1), then for each fixed $\ell$ we estimate $f_\ell$ as the variance of approximated $Z_\ell^m$, $m=-\ell,\dots,\ell$. In this case one can not expect good result for small $\ell$, since the estimator of the variance $f_\ell$ based on $2\ell+1$ values. The alternative method is based on the estimation of covariance function first then use the expansion (2.1) according to the Legendre polynomials for estimating $f_\ell$. The advantage of this later one is that there are many distances between pixels in which the estimation of the covariance is possible.

For further improvement of this computations we are going to apply some sampling theorems concerning on spherical harmonics and Legendre polynomials. We show this method through simulations.

In our simulations we consider random fields not only with zero mean but with $f_0=0$ as well. The reason is that we have only one realization and when we center the observation the sample mean contains a value of $Z_0^0$ hence $f_0$ can not be identified.

To the numerical approximation of the integral (2.2) denote by $t_1,t_2,\dots,t_n$ the nodes of the quadrature ($-1\le t_i\le 1$), and for a given $i$ let $(L_{1j}^i,L_{2j}^i)$, $j=1,\dots,N$, be pairs of pixels which have angular distance $t_i$. Considering the samples

$$X_1,X_2,\dots,X_N,\quad \text{where}\quad X_j=X(L_{1j}^i),$$

and

$$Y_1,Y_2,\dots,Y_N,\quad \text{where}\quad Y_j=X(L_{2j}^i),$$

we use the empirical covariance

$$\widehat{C}_i=\frac{1}{N}\sum_{j=1}^{N}X_jY_j$$

to estimate the value $C(t_i)$, $i=1,\dots,n$.

In the program we used only pixels located in the equatorial area (i.e. pixel centers with co-latitude $-\frac{1}{3}\le\cos\vartheta\le\frac{1}{3}$). E.g. in the case of $N_{side}=16$ these pixels determine nearly 9000 different values for $t$. In the equatorial zone each ring contains the same number of pixels ($4N_{side}$), moreover the pixel centers are equidistant located. In order to calculate the possible values of $t = \cos\gamma$ we considered the first pixel center on each ring in the north equatorial belt together with the pixel centers located on and below the actual ring. More precisely it is suffices to consider on each ring only the half of the pixels. After that for a given $t$ to collect the pixel-pairs having distance $t$ we can use the rotation symmetry. Depending on the location of the original pixel-pair $(L_1, L_2)$, which was used to compute $t$, there exist $4N_{side}$, $8N_{side}$ or $16N_{side}$ pairs having the given distance. If both of the pixels lie on the equator, or $\theta_1 = \pi - \theta_2$ and $\varphi_1 = \varphi_2$ (where $L_1 = (\theta_1, \varphi_1)$, $L_2 = (\theta_2, \varphi_2)$), that is the locations are symmetric to the equator, then the number of pairs is equal to $4N_{side}$. In the case of $\theta_1 = \theta_2 \neq \frac{\pi}{2}$ and in the case of $\theta_1 \neq \pi - \theta_2$ and $\varphi_1 = \varphi_2$, moreover if $\theta_1 = \pi - \theta_2$ and $\varphi_1 \neq \varphi_2$, there are $8N_{side}$ pairs. In all other cases there exist $16N_{side}$ pairs corresponding to the given distance.

By the numerical calculation of (2.2) using the Gaussian quadrature instead of the built-in Matlab function *trapz* enables a more efficient calculation, since these method requires much less evaluations of empirical covariances, however, this could be subject of further investigations.

**Test example 1.** (See Figure 1.) As a first example we considered the spatial process

$$X(L) = \sum_{\ell=1}^{100} \sqrt{f_\ell} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(L), \tag{4.1}$$

where $Z_\ell^m \sim \mathcal{N}(0, 2)$ are i.i.d. random numbers and

$$f_\ell = \frac{1}{(\ell(\ell+1)+4)^2}, \quad \ell = 1, 2, \ldots$$

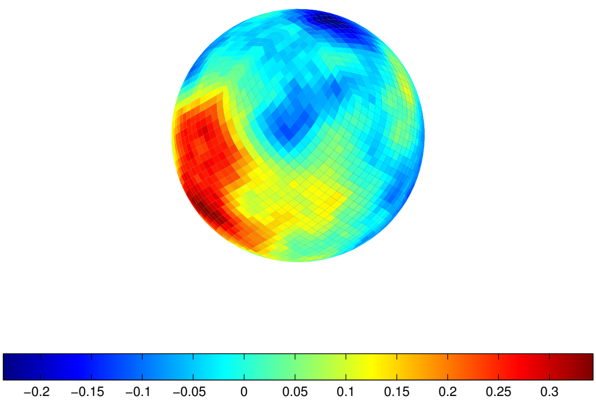


Figure 1: A random field described in Test example 1

By the discretization of the sphere we used $N_{side} = 16$ as resolution parameter, which results 3072 pixels located on 63 isolatitude rings.

The estimated and theoretical correlations can be seen on Figure 2 such that $f_0 = f_1 = 0$.

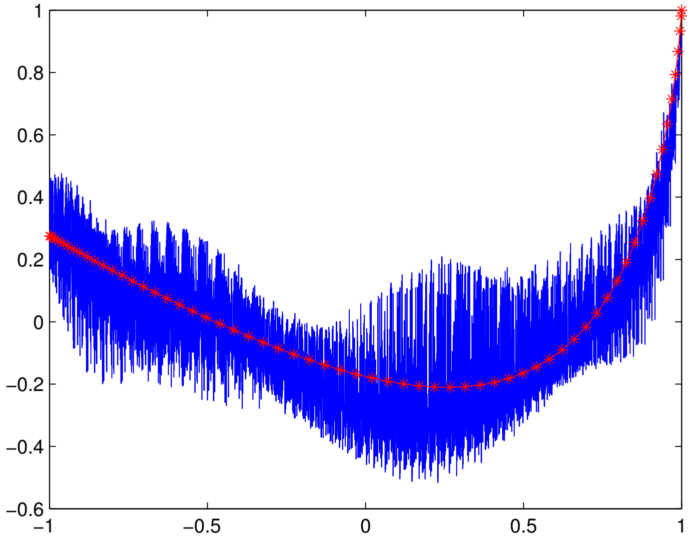


Figure 2: Estimated correlation in Test example 1

Let us denote by $\hat{f}_\ell$, $\ell = 1, \ldots, 100$ the estimated spectrum, then we obtained

$$\sum_{\ell=1}^{100} (f_\ell - \hat{f}_\ell)^2 \approx 1.93 \cdot 10^{-4}$$

and

$$\max_{1 \le \ell \le 100} |f_\ell - \hat{f}_\ell| = 4.2 \cdot 10^{-3}.$$

**Test example 2.** In the second example we investigated the field defined by the sum (4.1) taking

$$f_\ell = \frac{4\pi}{2\ell + 1} 0.8^\ell, \quad \ell = 1, 2, \ldots$$

The covariance is estimated from the generated field, and the theoretical correlation

$$C(\gamma) = \frac{1}{\sqrt{1 - 1.6 \cos\gamma + 0.8^2}} - 1;$$

are shown on Figure 3.

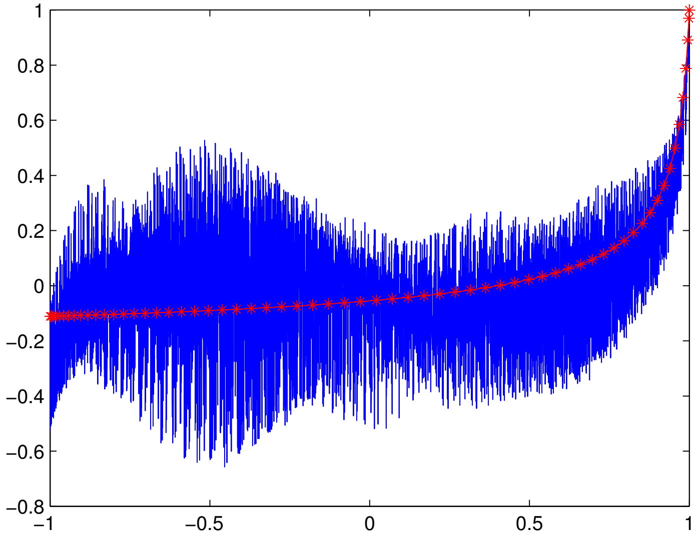


Figure 3: Estimated correlation in Test example 2

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