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On normals of manifolds in multidimensional projective space*

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Abstract

In the paper the regular hyper-zones in the multi-dimensional non-Euclidean space are discussed. The determined bijection between the normals of the first and second kind for the hyper-zone makes it possible to construct the bundle of normals of second-kind for the hyper-zone with assistance of certain bundle of normals of first-kind and vice versa. And hence the bundle of the normals of second-kind is constructed in the third-order differential neighbourhood of the forming element for hyper-zone. Research of hyper-zones and zones in multi-dimensional spaces takes up an important place in intensively developing geometry of manifolds in view of its applications to mechanics, theoretical physics, calculus of variations, methods of optimization.

Keywords: non-Euclidean space, regular hyper-zone, bundle of normals, bijection

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1. Introduction

In this article we analyze the theory of regular hyper-zone in the extended non-Euclidean space. We derive differential equations that define the hyper-zone SH_r

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with regards to a self-polar normalised frame of space ${}^{\lambda}S_n$. The tensors which determine the equipping planes in the third-order neighborhood of the hyper-zone are introduced. The bundles of the normals of the first and second kind are constructed by an inner invariant method in the third-order differential neighbourhood of the forming element for hyper-zone. The bijection between the normals of the first and second kind for the hyper-zone SH_r is determined.

The concept of zone was introduced by W. Blaschke [1]. V. Wagner [7] was the first who proposed to consider the surface equipped with the field of tangent hyper-planes in the n-dimensional centro-affine space.

We apply the group-theoretical method for research in differential geometry developed by professor G.F. Laptev [4]. At present the method of Laptev remains the most efficient way of research for manifolds, immersed in generalized spaces. We use results obtained in the article [3].

For the past years the methods of generalizations of Theory of regular and singular hyper-zones (zones) with assistance of the Theory of distributions in multidimensional affine, projective spaces and in spaces with projective connections were studied by A.V. Stolyarov, Y.I. Popov and M.M. Pohila. In this article we analyze the theory of regular hyper-zone in the extended non-Euclidean space. We derive differential equations that define the hyper-zone SH_r with regards to a self-polar normalised frame of space ${}^{\lambda}S_n$. The tensors which determine the equipping planes in the third-order neighborhood of the hyper-zone are introduced. The bundles of the normals of the first and second kind are constructed by an inner invariant method in the third-order differential neighbourhood of the forming element for hyper-zone. The bijection between the normals of the first and second kind for the hyper-zone SH_r is determined.

Before M. Grebenyuk and J. Mikeš in the article [2] discussed the theory of the linear distribution in affine space. The bundles of the projective normals of the first kind for the equipping distributions are constructed by an inner invariant method in second and third differential neighbourhoods of the forming element. In the article we apply the group-theoretical method for research in differential geometry developed by G.F. Laptev [4]. At present the method of Laptev remains the most efficient way of research for manifolds, immersed in generalized spaces. We use results obtained in the article [3].

2. Definition of the hyper-zone in the extended non-Euclidean space

Let a non-degenerated hyper-quadric be given in a projective n-dimensional space P_n as

$$q'_{IJ}x^I x^J = 0, \quad q'_{IJ} = q'_{JI}, \quad \det \|q'_{IJ}\| \neq 0, \qquad I, J = 0, 1, \dots, n,$$

where the smallest number of the coefficients of the same sign is equal to λ. Thus, it is possible to determine a subgroup of collineations for space P_n, which are preserving this hyper-quadric and, hence, it is possible to introduce a projective metrics.

Let us name the obtained in this way metric space with this fundamental group as the extended non-Euclidian space ${}^{\lambda}S_n$ with index λ [5], and the corresponding hyper-quadric as the absolute of the space ${}^{\lambda}S_n$.

Let us consider a plane element (A, τ) in the space ${}^{\lambda}S_n$ which is composed of a point A and a hyper-plane τ, where point A belongs to plane τ.

Definition 2.1. Suppose that the point A defines an r-dimensional surface V_r and the hyper-plane $\tau(A)$ is tangent to the surface V_r in the corresponding points $A \in V_r$. Then the r-parametric manifold of the plane elements (A, τ) is called *r-parametric hyper-zone* $SH_r \subset {}^{\lambda}S_n$. The surface V_r is called the base surface and the hyper-planes $\tau(A)$ are called the principal tangent hyper-planes to the hyper-zone SH_r.

Definition 2.2. The characteristic plane $X_{n-r-1}(A)$ for the tangent hyper-plane $\tau = \tau(u^1, \ldots, u^r)$ is called the *characteristic plane* for the hyper-zones SH_r at the point $A(u^1, \ldots, u^r)$.

Definition 2.3. The hyper-zone SH_r is called *regular* if the characteristic plane $X_{n-r-1}(A)$ and the tangent plane $T_r(A)$ for directing surface V_r for hyper-zone SH_r at each point $A \in V_r$ have no common straight lines.

The regular hyper-zone SH_r in a self-polar normalized basis $\{A_0, A_1, \ldots, A_n\}$ in the space ${}^{\lambda}S_n$ is defined as follows:

$$\omega_o^n = 0, \quad \omega_o^\alpha = 0, \quad \omega_\alpha^n = 0, \quad \omega_n^o = 0, \quad \omega_n^\alpha = 0, \quad \omega_\alpha^o = 0,$$

$$\omega_i^n = a_{ij}\omega^j, \quad \omega_\alpha^i = b_{\alpha j}^i\omega^j, \quad \omega_i^\alpha = b_{ij}^\alpha\omega^j, \quad \omega_i^o = -\varepsilon_{oi}\omega^i, \quad \omega_n^i = \varepsilon_{in}a_{ij}\omega^j,$$

$$\nabla a_{ij} = -a_{ij}\omega_n^n - a_{ijk}\omega^k, \quad \nabla b_{\alpha j}^i = b_{\alpha jk}^i\omega^k, \quad \nabla b_{ij}^\alpha = b_{ijk}^\alpha\omega^k,$$

where

$$b_{\alpha j}^i a_{i\ell} = b_{\alpha\ell}^i a_{ij}, \quad b^\alpha ik = -\varepsilon_{\alpha i} b_a^{ij} a_{jk}, \quad b_{\alpha k}^i = b_\alpha^{ij} a_{jk},$$

and functions $b_{\alpha jk}^i$ are symmetric according to indices j and k.

Systems of objects

$$\Gamma_2 = \{a_{ij}, b_{\alpha j}^i\}, \quad \Gamma_3 = \{\Gamma_2, a_{ijk}, b_{\alpha jk}^i\}$$

make up fundamental objects of second and third orders respectively for hyper-zone $SH_r \subset {}^{\lambda}S_n$.

3. Canonical bundle of projective normals for the hyper-zone

With the help of the components of fundamental geometric object of the third order for hyper-zone $SH_r \subset {}^{\lambda}S_n$ let us construct the quantities

$$d_i = \frac{1}{r+2}\, a_{ijk}a^{jk}, \quad \nabla_\delta d_i = 0,$$

$$d^i = \frac{1}{r+2}\, a^{ijk}a_{jk}, \quad \nabla_\delta d^i = d^i\pi_n^n.$$

The tensors d_i and d^i define dual equipping planes in the third-order neighborhood of the hyper-zone SH_r

$$E_{r-1} \equiv [M_i] = [A_i + d_i A_o], \quad E_{n-r} \equiv [\sigma^i] = [\tau^i + d^i \tau^n].$$

Using the Darboux tensor

$$\mathcal{L}_{ijk} = a_{ijk} - a_{(ij} d_{k)},$$

one builds the symmetric tensor

$$L_{ij} = a^{k\ell} a^{mp} \mathcal{L}_{ikm} \mathcal{L}_{j\ell p}, \qquad \nabla_\delta L_{ij} = 0,$$

which is non-degenerate in general case.

Let us consider a field of straight lines associated with the hyper-zone SH_r

$$h(A_o) = [A_o, P], \qquad P = A_n + x^i A_i + x^\alpha A_\alpha,$$

where each line passes through the respective point A of the directing surface V_r and do not belong to the tangent hyper-plane $\tau(A_o)$.

Let us require that straight line $h = [A_o, P]$ is an invariant line, i.e. $\delta h = \theta h$. The last condition is equivalent to the differential equations:

$$\nabla_\delta \chi^\alpha = \chi^\alpha \pi_n^n \qquad \text{and} \qquad \nabla_\delta \chi^i = \chi^i \pi_n^n.$$

First equations are realized on the condition that $x^\alpha = B^\alpha$, and second equations have two solutions:

$$x^i = -d^i, \quad x^i = B^i.$$

Hence, the system of the differential equations has a general solution of the following form:

$$x^i = -d^i + \sigma(B^i + d^i),$$

where σ is the absolute invariant.

Thus, we obtain the bundle of straight lines, which is associated with the hyper-zone SH_r by inner invariant method:

$$h(\sigma) = [A_o, P(\sigma)] = [A_o, A_n + \{(\sigma - 1)d^i + \sigma B^i\} A_i + B^\alpha A_\alpha],$$

where σ is the absolute invariant.

The constructed projective invariant bundle of straight lines makes it possible to construct the invariant bundle of first-kind normals E_{n-r}, which is associated by the inner method with the hyper-zone SH_r in the differential neighborhood of the third order of its generatrix element.

Consequently, it is possible to represent each invariant first kind normal $E_{n-r}(A_o)$ as the $(n-r)$-plane that encloses the invariant straight line $h(A_o)$ and the characteristic $X_{n-r-1}(A_o)$ for hyper-zone SH_r [6].

$$E_{n-r}(\sigma) \stackrel{def}{=} [X_{n-r-1}(A_o);\ A_n + \{(\sigma - 1)d^i + \sigma B^i\}\ A_i + B^\alpha A_\alpha,$$

where σ is the absolute invariant.

4. Bijection between first- and second-kind normals of the hyper-zone SH_r

Let us introduce the correspondence between the normals of the first- and second-kind for the hyper-zone SH_r. For that, let us construct a tensor:

$$P_i = -a_{ij}\nu^j + d_i, \quad \nabla_\delta P_j = 0 \tag{4.1}$$

where ν^j is the tensor satisfying the condition $\nabla_\delta \nu^j = \nu^j \pi_n^n$.

The tensor P_i defines the normal of second-kind for hyper-zone SH_r that is determined by the points

$$M_i = A_i + \chi_i A_o, \quad \nabla_\delta \chi_i = 0.$$

Further, the tensor ν^j can be represented using the components of the tensor P_i as follows

$$\nu^j = -P_i a^{ij} + d^j.$$

Therefore, the bijection between the normals of the first- and second-kind for the hyper-zone SH_r is obtained using the relations (4.1). The constructed bijection makes it possible to determine the bundle of second-kind normals, using the bundle of first-kind normals and vice versa. Therefore, we got constructed the bundle of second-kind normals, which is associated by the inner method with the hyper-zone SH_r in the differential neighborhood of the third order of its generating element. So true the following theorem.

Theorem 4.1. *Tensor P_i defines the bijection between the normals of the first- and second-kind for the hyper-zone SH_r.*

Finally, we get the theorem.

Theorem 4.2. *Tensor $\nu^j = -P_i a^{ij} + d^i$ defines the bundle of second- kind normals, which is associated by inner method with the hyper-zone SH_r in the differential neighborhood of the third order of its generating element.*

References

[1] Blaschke, W., *Differential geometry*, I. OSTI, M., L., (1935).

[2] Grebenyuk, M., Mikeš, J., Equipping Distributions for Linear Distribution, *Acta Palacky Univ. Olomuc., Mathematica*, 46 (2007) 35–42.

[3] Kosarenko, M. F., Fields of the geometrical objects for regular hyper-zone $SH_r \subset {}^\lambda S_n$, (Russian), Differential geometry of manifolds of figures. *Interuniversity subject collection of scientific works. Kaliningrad Univ.*, 13 (1982) 38–44.

[4] Laptev, G. F., Differential geometry of embedded manifolds: group-theoretical method for research in differential geometry, *Proc. of the Moscow Math. Soc.*, 2 (1953) 275–382.

[5] Rozenfeld, B. A., Non-Euclidean geometries, *Gostechizdat, Moscow*, (1955).

[6] Stoljarov, A. V., Differential geometry of zones, *Itogi Nauki Tekh., Ser. Probl. Geom.*, 10 (1978) 25–54.

[7] Wagner, V., Field theory for local hyper-zones, *Works of seminar on vector and tensor analysis*, 8 (1950) 197-272.