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Divisible and cancellable subsets of groupoids*

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Abstract

In this paper, after listing some basic facts on groupoids, we establish several simple consequences and equivalents of the following basic definitions and their obvious counterparts.

For some $n \in \mathbb{N}$, a subset U of a groupoid X is called

(1) n-cancellable if $nx = ny$ implies $x = y$ for all $x, y \in U$,

(2) n-divisible if for each $x \in U$ there exists $y \in U$ such that $x = ny$.

Moreover, for some $A \subset \mathbb{N}$, the set U is called A-divisible (A-cancellable) if it is n-divisible (n-cancellable) for all $n \in A$.

Our main tools here are the sets $n^{-1}x = \{y \in X : x = ny\}$ satisfying $n(n^{-1}x) \subset \{x\} \subset n^{-1}(nx)$ for all $n \in \mathbb{N}$ and $x \in X$. They can be used to briefly reformulate properties (1) and (2), and naturally turn a uniquely $\mathbb{N}$-divisible commutative group into a vector space over $\mathbb{Q}$.

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1. A few basic facts on groupoids

Definition 1.1. If X is a set and $+$ is a function of X^2 to X, then the function $+$ is called a *binary operation* on X, and the ordered pair $X(+) = (X, +)$ is called a *groupoid*.

Remark 1.2. In this case, we may simply write $x + y$ in place of $+(x, y)$ for all $x, y \in X$. Moreover, we may also simply write X in place of $X(+)$.

Instead of groupoids, it is more customary to consider only *semigroups* (associative grupoids) or even *monoids* (semigroups with zero). However, several definitions on semigroups can be naturally extended to groupoids.

Definition 1.3. If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$, we define

$$nx = x \quad \text{if} \quad n = 1 \qquad \text{and} \qquad nx = (n-1)x + x \quad \text{if} \quad n > 1.$$

Now, by induction, we can easily prove the following two basic theorems.

Theorem 1.4. *If X is a semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have*

(1) $(m + n)x = mx + nx$,

(2) $(nm)x = n(mx)$.

Proof. To prove (2), note that if $(nm)x = n(mx)$ holds for some $n \in \mathbb{N}$, then by (1) we also have

$$((n + 1)m)x = (nm + m)x = (nm)x + mx = n(mx) + mx = (n + 1)(mx). \qquad \square$$

Theorem 1.5. *If X is a semigroup, then for any $m, n \in \mathbb{N}$ and $x, y \in X$, with $x + y = y + x$, we have*

(1) $mx + ny = ny + mx$,

(2) $n(x + y) = nx + ny$.

Proof. To prove (1), note that if $x + ny = ny + x$ holds for some $n \in \mathbb{N}$, then we also have

$$x + (n + 1)y = x + ny + y = ny + x + y = ny + y + x = (n + 1)y + x.$$

While, to prove (2), note that if $n(x+y) = nx+ny$ holds for some $n \in \mathbb{N}$, then by (1) we also have

$$\begin{aligned}(n + 1)(x + y) &= n(x + y) + x + y = nx + ny + x + y = \\ &= nx + x + ny + y = (n + 1)x + (n + 1)y. \qquad \square\end{aligned}$$

Definition 1.6. If in particular X is a groupoid with zero, then we also define $0x = 0$ for all $x \in X$.

Moreover, if more specially X is a *group*, then we also define $(-n)x = n(-x)$ for all $x \in X$ and $n \in \mathbb{N}$.

Lemma 1.7. *If X is a group, then for any $x \in X$ and $n \in \mathbb{N}$ we also have $(-n)x = -(nx)$.*

Proof. By using $-x + x = 0 = x + (-x)$ and Theorem 1.5, we can at once see that $n(-x) + nx = n(-x + x) = n0 = 0$. Therefore, $n(-x) = -(nx)$, and thus the required equality is also true. □

Now, we can also easily prove the following counterparts of Theorems 1.4 and 1.5.

Theorem 1.8. *If X is a group, then for any $x \in X$ and $k, l \in \mathbb{Z}$ we have*
(1) $(kl)x = k(lx)$,
(2) $(k + l)x = kx + lx$.

Theorem 1.9. *If X is a group, then for any $k, l \in \mathbb{Z}$ and $x, y \in X$, with $x + y = y + x$, we have*
(1) $kx + ly = ly + kx$,
(2) $k(x + y) = kx + ky$.

Proof. To prove (2), note that by Lemma 1.7, Theorem 1.5 and assertion (1) we have

$$\begin{aligned}(-n)(x + y) &= -\big(n(x + y)\big) = -(nx + ny)\\ &= -(ny) + \big(-(nx)\big) = (-n)y + (-n)x = (-n)x + (-n)y\end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, $0(x + y) = 0 = 0x + 0y$ also holds. □

Remark 1.10. The latter two theorems show that a commutative group X is already a *module* over the ring $\mathbb{Z}$ of all integers.

2. Operations with subsets of groupoids

Definition 2.1. If X is a groupoid with zero, then for any $U \subset X$ we define

$$U_0 = U \cup \{0\} \quad \text{if} \quad 0 \notin U \qquad \text{and} \qquad U_0 = U \setminus \{0\} \quad \text{if} \quad 0 \in U.$$

Remark 2.2. In the sequel, this particular unary operation will mainly be applied to the subsets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ of the additive group $\mathbb{R}$ of all real numbers.

Definition 2.3. If X is a groupoid, then for any $A \subset \mathbb{N}$, and $U, V \subset X$ we define

$$AU = \big\{nu : n \in A, u \in U\big\} \quad \text{and} \quad U + V = \big\{u + v : u \in U, v \in V\big\}.$$

Remark 2.4. Now, by identifying singletons with their elements, we may simply write $nU = \{n\}U$, $Au = A\{u\}$, $u + V = \{u\} + V$, and $U + v = U + \{u\}$ for all $n \in \mathbb{N}$ and $u, v \in X$.

The notation nU may cause some confusions since in general we only have $nU \subset (n - 1)U + U$ for all $n > 1$. However, assertions 1.4(1),(2) and 1.5(1) can be generalized to sets.

Remark 2.5. If in particular, X is a group, then we may quite similarly define AU for all $A \subset \mathbb{Z}$ and $U \subset X$.

Moreover, we may also naturally define $-U = (-1)U$ and $U - V = U + (-V)$ for all $V \subset X$. However, thus we have $U - U = \{0\}$ if and only if U is a singleton.

Remark 2.6. Moreover, if more specially if X is a *vector space* over K, then we may also quite similarly define AU for all $A \subset K$ and $U \subset X$.

Thus, only two axioms of a vector space may fail to hold for $\mathcal{P}(X)$. Namely, in general, we only have $(\lambda + \mu)U \subset \lambda U + \mu U$ for all $\lambda, \mu \in K$.

The corresponding elementwise operations with subsets of various algebraic structures allow of some more concise treatments of several basic theorems on substructures of these structures.

Remark 2.7. For instance, a subset U of a groupoid X is called a *subgroupoid* of X if U is itself a groupoid with respect to the restriction of the addition on X to $U \times U$.

Thus, U is a subgroupoid of X if and only if U is *superadditive* in the sense $U + U \subset U$. Moreover, if U is a subgroupoid of X, then U is in particular $\mathbb{N}$-*superhomogeneous* in the sense that $\mathbb{N}U \subset U$.

Concerning subgroups, we can prove some more interesting theorems.

Theorem 2.8. *If X is a group, then for a nonvoid subset U of X the following assertions are equivalent:*

(1) *U is a subgroup of X,*

(2) *$-U \subset U$ and $U + U \subset U$,*

(3) *$U - U \subset U$.*

Remark 2.9. Note that if U is a subset of a group X such that $-U \subset U$, then U is already *symmetric* in the sense that $-U = U$.

While, if U is a subset of a groupoid X with zero such that $U + U \subset U$ and $0 \in U$, then U is already *idempotent* in the sense that $U + U = U$.

Therefore, as an immediate consequence of Theorem 2.8, we can also state

Corollary 2.10. *A nonvoid subset U of a group X is a subgroup of X if and only if it is symmetric and idempotent.*

In addition to Theorem 2.8, we can also easily prove the following

Theorem 2.11. *If X is a group, then for any two symmetric subsets U and V of X the following assertions are equivalent:*

(1) *$U + V = V + U$,*

(2) *$U + V$ is symmetric.*

Proof. If (1) holds, then $-(U + V) = -V + (-U) = V + U = U + V$, and thus (2) also holds.

While, if (2) holds, then $U + V = -(U + V) = -V + (-U) = V + U$, and thus (1) also holds. $\square$

Remark 2.12. If U and V are idempotent subsets of a semigroup X such that (1) holds, then

$$U + V + U + V = U + V + V + U = U + V + U = U + U + V = U + V,$$

and thus $U + V$ is also an idempotent subset of X.

Therefore, as an immediate consequence of Theorem 2.11 and Corollary 2.10, we can also state

Theorem 2.13. *If X is a group, then for any two subgroups U and V of X the following assertions are equivalent:*

(1) $U + V = V + U$,

(2) $U + V$ *is a subgroup of* X.

Hence, it is clear that in particular we also have the following

Corollary 2.14. *If U and V are commuting subgroups of a group X, then $U + V$ is the smallest subgroup of X containing both U and V.*

Remark 2.15. In the standard textbooks, Theorem 2.13, or its corollary, is usually proved directly without using Theorems 2.8 and 2.11. (See, for instance, Sott [13, p. 18] and Burton [4, p. 118].)

3. Direct sums of subsets of groupoids

Analogously to Fuchs [6, p. 3.15], we may naturally introduce the following

Definition 3.1. If U, V and W are subsets of a groupoid X such that for every $x \in W$ there exists a unique pair $(u_x, v_x) \in U \times V$ such that

$$x = u_x + v_x,$$

then we say that W is the *direct sum* of U and V, and we write $W = U \oplus V$.

Remark 3.2. Thus, in particular we have $W = U + V$. Hence, if in addition X has a zero such that $0 \in V$, we can infer that $U \subset W$.

Moreover, in this particular case for any $x \in U$ we have $x = x + 0$. Hence, by using the unicity of u_x and v_x, we can infer that $u_x = x$ and $v_x = 0$.

Remark 3.3. Therefore, if $W = U \oplus V$ and in particular X has a zero such that $0 \in U \cap V$, then in addition to $W = U + V$ we can also state that $U \cup V \subset W$ and $U \cap V = \{0\}$.

Namely, by Remark 3.2 and its dual, we have $U \subset W$ and $V \subset W$, and thus $U \cup V \subset W$. Moreover, if $x \in U \cap V$, i.e., $x \in U$ and $x \in V$, then we have $v_x = 0$ and $u_x = 0$, and thus $x = u_x + v_x = 0$.

In this respect, we can also easily prove the following

Theorem 3.4. *If U and V are subgroups of a monoid X, with $0 \in U \cap V$, then the following assertions are equivalent:*

(1) $X = U \oplus V$;

(2) $X = U + V$ *and* $U \cap V = \{0\}$.

Proof. If $x \in X$ such that $x = u_1 + v_1$ and $x = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$, then $u_1 + v_1 = u_2 + v_2$, and thus $-u_2 + u_1 = v_2 - v_1$. Moreover, we also have $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$. Hence, if the second part of (2) holds, we can infer that $-u_2 + u_1 = 0$ and $v_2 - v_1 = 0$. Therefore, $u_1 = u_2$, and $v_1 = v_2$ also hold. □

Remark 3.5. Note that if U and V are subgroups of a monoid X, with $0 \in U \cap V$, such that $X = U + V$, then for any $x \in X$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Hence, by taking $y = -v - u$, we can see that $x + y = 0$ and $y + x = 0$. Therefore, $-x = y$, and thus X is also a group.

Remark 3.6. Note that if G is a group, then the Descartes product $X = G \times G$, with the coordinatewise addition, is also a group. Moreover,

$$U = \big\{(x, 0) : x \in G\big\} \quad \text{and} \quad V = \big\{(0, y) : y \in G\big\}$$

are subgroups of X such that $X = U + V$ and $U \cap V = \{(0, 0)\}$. Therefore, by Theorem 3.4, we also have $X = U \oplus V$.

Furthermore, it is also worth noticing that the sets U and V are elementwise commuting in the sense that $u + v = v + u$ for all $u \in U$ and $v \in V$.

The importance of elementwise commuting sets is apparent from the following

Theorem 3.7. *If U and V are elementwise commuting subgroupoids of a semigroup X such that $X = U \oplus V$, then the mappings*

$$x \mapsto u_x \quad \textit{and} \quad x \mapsto v_x,$$

where $x \in X$, are additive. Thus, in particular, they are $\mathbb{N}$-homogeneous.

Proof. If $x, y \in X$, then by the assumed associativity and commutativity properties of the addition in X we have

$$x + y = (u_x + v_x) + (u_y + v_y) = (u_x + u_y) + (v_x + v_y).$$

Therefore, since $u_x + u_y \in U$ and $v_x + v_y \in V$, the equalities

$$u_{x+y} = u_x + u_y \quad \text{and} \quad v_{x+y} = v_x + v_y$$

are also true.

Moreover, by induction, it can be easily seen that if f is an additive function of one groupoid X to another Y, then $f(nx) = nf(x)$ for all $n \in \mathbb{N}$ and $x \in X$. □

Remark 3.8. Note that if in particular X has a zero such that $0 \in V$, then by Remark 3.2 the mapping $x \mapsto u_x$, where $x \in X$, is idempotent. Moreover, if $0 \in U$ also holds, then $u_0 = 0$. Thus, the above mapping is also zero-homogeneous.

Remark 3.9. In this respect, it is also worth noticing that if in particular X is a monoid, and U and V are subgroups of X, with $0 \in U \cap V$, then by Remark 3.5 X is also a group, and thus the mappings considered in Theorem 3.7 are actually $\mathbb{Z}$-homogeneous.

Remark 3.10. If in particular X is a vector space, then by using Zorn's lemma [14, p. 38] it can be shown that for each subspace U of X there exists a subspace V of X such that $X = U \oplus V$.

In the standard textbooks, this fundamental decomposition theorem is usually proved with the help of Hamel bases. (See, for instance, Cotlar and Cignoli [5, p. 15] and Taylor and Lay [14, p. 43].)

4. Some further results on elementwise commuting sets

The importance of elementwise commuting sets is also apparent from the following

Theorem 4.1. *If U and V are elementwise commuting, comutative subsets of a semigroup X, then $U + V$ is also commutative.*

Proof. Namely, if $x, y \in U + V$, then there exist $u, \omega \in U$ and $v, w \in V$ such that $x = u + v$ and $y = \omega + w$. Hence, we can already see that

$$x + y = u + v + \omega + w = u + \omega + v + w = \omega + u + w + v = \omega + w + u + v = y + x.$$

Therefore, the required assertion is also true. □

Remark 4.2. Conversely, we can also easily note that if U and V are subsets of a groupoid X such that $U + V$ is commutative and $U \cup V \subset U + V$, then U and V are commutative and elementwise commuting.

Therefore, as an immediate consequence of Theorem 4.1, we can also state

Corollary 4.3. *If U and V are subsets of monoid X such that $0 \in U \cap V$, then the following assertions are equivalent:*

(1) *$U + V$ is commutative,*

(2) *U and V are commutative and elementwise commuting.*

Remark 4.4. Note that if U and V are elementwise commuting subsets of a groupoid X, then we have not only $U + V = V + U$, but also $u + V = V + u$ and $U + v = v + U$ for all $u \in U$ and $v \in V$.

Therefore, it is of some interest to note that we also have the following

Theorem 4.5. *If U and V are subsets of a groupoid X such that $U+V=U\oplus V$, then the following assertions are equivalent:*
(1) *U and V are elementwise commuting,*
(2) *$u+V=V+u$ and $v+U=U+v$ for all $u\in U$ and $v\in V$,*
(3) *$u+V\subset V+u$ and $v+U\subset U+v$ for all $u\in U$ and $v\in V$,*
(4) *$V+u\subset u+V$ and $U+v\subset v+U$ for all $u\in U$ and $v\in V$.*

Proof. Namely, if for instance (3) holds, then for any $u\in U$ and $v\in V$ we have $u+v\in u+V\subset V+u$. Therefore, there exists $w\in V$ such that $u+v=w+u$. Moreover, again by (3), we can see that $w+u\in w+U\subset U+w$. Therefore, there exists $\omega\in U$ such that $w+u=\omega+w$. Thus, we also have $u+v=\omega+w$. Hence, by using that $U+V=U\oplus V$, we can infer that $u=\omega$ and $v=w$. Therefore, $u+v=v+u$, and thus (1) is also true. □

Remark 4.6. In this respect, it is also worth noticing that if U is a subset and V is a subgroup of a monoid X, then the following assertions are also equivalent:
(1) $U+v=v+U$ for all $v\in V$,
(2) $U+v\subset v+U$ for all $v\in V$,
(3) $v+U\subset U+v$ for all $v\in V$.

Namely, if for instance (2) holds, then we have

$$v+U=v+U+0=v+U+(-v)+v\subset v+(-v)+U+v=0+U+v=U+v$$

for all $v\in V$, and thus (1) also holds.

Concerning elementwise commuting sets, by Theorems 1.5 and 1.9, we can at once state the following two theorems.

Theorem 4.7. *If U and V are elementwise commuting sets of a semigroup X, then the sets $\mathbb{N}U$ and $\mathbb{N}V$ are also also elementwise commuting.*

Theorem 4.8. *If U and V are elementwise commuting subsets of a group X, then the sets $\mathbb{Z}U$ and $\mathbb{Z}V$ are also also elementwise commuting.*

Moreover, concerning elementwise commuting sets, we can also easily prove

Theorem 4.9. *If U and V are elementwise commuting subsets of a semigroup X such that U is commutative, then U and $U+V$ are also elementwise commuting.*

Proof. Suppose that $x\in U$ and $y\in U+V$. Then, there exist $u\in U$ and $v\in V$ such that $y=u+v$. Moreover, by the assumed commutativity properties of U and V, we have

$$x+y=x+u+v=u+x+v=u+v+x=y+x.$$

Therefore, the required assertion is also true. □

Remark 4.10. The importance of elementwise commuting subsets will also be well shown by the forthcoming theorems of Section 10.

5. Divisible and cancellable subsets of groupoids

Analogously to Hall [10, p. 197], Fuchs [6, p. 58] and Scott [13, p. 95], we may naturally introduce the following

Definition 5.1. A subset U of a groupoid X is called *n-divisible*, for some $n \in \mathbb{N}$, if $U \subset nU$.

Now, the subset U may also be naturally called *A-divisible*, for some $A \subset \mathbb{N}$, if it is n-divisible for all $n \in A$.

Remark 5.2. Thus, U is n-divisible if and only if it is *n-subhomogeneous.* That is, for each $x \in U$ there exists $y \in U$ such that $x = ny$.

Therefore, the set U may be naturally called *uniquely n-divisible* if for each $x \in U$ there exists a unique $y \in U$ such that $x = ny$.

Moreover, the subset U may also be naturally called *uniquely A-divisible* if it is uniquely n-divisible for all $n \in A$.

Now, in addition to Definition 5.1, we may also naturally introduce the following definition which has also been utilized in [8].

Definition 5.3. A subset U of a groupoid X is called *n-cancellable*, for some $n \in \mathbb{N}$, if $nx = ny$ implies $x = y$ for all $x, y \in U$.

Now, the set U may also be naturally called *A-cancellable*, for some $A \subset \mathbb{N}$, if it is n-cancellable for all $n \in A$.

Remark 5.4. Thus, if U is both n-divisible and n-cancellable, then U is already uniquely n-divisible.

Namely, if $x \in U$ such that $x = ny_1$ and $x = ny_2$ for some $y_1, y_2 \in U$, then we also have $ny_1 = ny_2$, and hence $y_1 = y_2$.

Remark 5.5. Moreover, by using some obvious analogues of Definitions 5.1 and 5.3, we can also see that if U is a both k-divisible and k-cancellable subset of a group X, for some $k \in \mathbb{Z}$, then U is already uniquely k-divisible.

In this respect, it is worth noticing that the following two theorems are also true.

Theorem 5.6. *If U is an n-superhomogeneous subset of a groupoid X, for some $n \in \mathbb{N}$, then the following assertions are equivalent:*

(1) *U is uniquely n-divisible,*

(2) *U is both n-divisible and n-cancellable.*

Proof. Namely, if (1) holds and $x, y \in U$ such that $nx = ny$, then because of $nx \in U$ and (1) we also have $x = y$. Therefore, U is n-cancellable, and thus (2) also holds. The converse implication (2) $\Longrightarrow$ (1) has been proved in Remark 5.4. □

Theorem 5.7. *If U is a k-superhomogeneous subset of a group X, for some $k \in \mathbb{Z}$, then following assertions are equivalent:*

(1) *U is uniquely k-divisible,*

(2) *U is both k-divisible and k-cancellable.*

By using the corresponding definitions and Theorems 1.4 and 1.8, we can easily prove the following two theorems.

Theorem 5.8. *If U is an n-divisible subset of a semigroup X, for some $n \in N$, and $p, q \in \mathbb{N}$ such that $n = pq$ and U is q-superhomogeneous, then U is also p-divisible.*

Proof. If $x \in U$, then by the n-divisibility of U there exists $y \in U$ such that $x = ny$. Now, by using Theorem 1.4, we can see that $x = ny = (pq)y = p(qy)$. Hence, because of $qy \in U$, it is clear that U is also p-divisible. $\square$

Theorem 5.9. *If U is an k-divisible subset of a semigroup X, for some $k \in Z$, and $p, q \in \mathbb{Z}$ such that $k = pq$ and U is q-superhomogeneous, then U is also p-divisible.*

In addition to the latter two theorems, it is also worth proving the following

Theorem 5.10. *For a subset U of a monoid X, the following assertions are equivalent:*

(1) $U \subset \{0\}$,

(2) U *is* 0*-divisible,*

(3) U *is* $\mathbb{N}_0$*-divisible.*

By using the corresponding definitions and Theorems 1.4 and 1.8, we can also easily prove the following counterparts of Theorems 5.8, 5.9 and 5.10.

Theorem 5.11. *If U is an m-superhomogeneous, both n- and m-cancellable subset of a semigroup X, for some $m, n \in \mathbb{N}$, then U is also nm-cancellable.*

Proof. If $x, y \in U$ such that $(nm)x = (nm)y$, then by Theorem 1.4 we also have $n(mx) = n(my)$. Hence, by using the n-cancelability of U, and the fact that $mx, my \in U$, we can infer that $mx = my$. Now, by the m-cancelability of U, we can see that $x = y$. Therefore, U is also nm-cancellable. $\square$

Theorem 5.12. *If U is an l-superhomogeneous, both k- and l-cancellable subset of a group X, for some $k, l \in \mathbb{N}$, then U is also kl-cancellable.*

Theorem 5.13. *For a subset U of a monoid X, the following assertions are equivalent:*

(1) $\operatorname{card}(U) \leq 1$,

(2) U *is* 0*-cancellable,*

(3) U *is* $\mathbb{N}_0$*-cancellable.*

In addition to Theorems 5.8 and 5.9, we can also prove the following two theorems.

Theorem 5.14. *If U is a uniquely n-divisible, n-superhomogeneous subset of a semigroup X for some $n \in N$, and $p, q \in \mathbb{N}$ such that $n = pq$ and U is q-superhomogeneous, then U is also uniquely p-divisible.*

Proof. By Theorem 5.8 and Remark 5.4, we need only show that now U is also p-cancellable.

For this, note that if $x, y \in U$ such that $px = py$, then by Theorem 1.4 we also have $nx = (qp)x = q(px) = q(py) = (qp)x = ny$. Moreover, by Theorem 5.6, U is now n-cancellable. Therefore, we necessarily have $x = y$. □

Theorem 5.15. *If U is a uniquely k-divisible, k-superhomogeneous subset of a group X, for some $k \in Z$, and $p, q \in \mathbb{Z}$ such that $n = pq$ and U is q-superhomogeneous, then U is also uniquely p-divisible.*

Remark 5.16. Note that in assertion (3) of Theorem 5.10 we may also write "uniquely $\mathbb{N}_0$-divisible" instead of "$\mathbb{N}_0$-divisible".

6. Some further results on divisible and cancellable sets

Theorem 6.1. *If U is a k-divisible, symmetric subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$-divisible.*

Proof. If $x \in U$, then by the k-divisibility of U there exists $y \in U$ such that $x = ky$. Now, by using Theorem 1.8, we can see that

$$x = ky = \big((-k)(-1)\big)y = (-k)\big((-1)y\big) = (-k)(-y).$$

Hence, since now we also have $-y \in -U = U$, it is clear that U is also $-k$-divisible. □

From this theorem, it is clear that in particular we also have

Corollary 6.2. *If U is an $\mathbb{N}$-divisible, symmetric subset of a group X, then U is $\mathbb{Z}_0$-divisible.*

Analogously to Theorem 6.1, we can also easily prove the following

Theorem 6.3. *If U is a k-cancellable subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$-cancellable.*

Proof. If $x, y \in U$ such that $(-k)x = (-k)y$, then by Theorem 1.8 we also have

$$kx = \big((-1)(-k)\big)x = (-1)\big((-k)x\big) = (-1)\big((-k)y\big) = \big((-1)(-k)\big)y = ky.$$

Hence, by the assumption, it follows that $x = y$, and thus the required assertion is also true. □

From this theorem, it is clear that in particular we also have

Corollary 6.4. *If U is an $\mathbb{N}$-cancellable subset of a group X, then U is also $\mathbb{Z}_0$-cancellable.*

Now, as an immediate consequence of Theorems 6.1 and 6.3 and Remark 5.5, we can also state

Theorem 6.5. *If U is a uniquely k-divisible, symmetric subset of a group X, for some $k \in \mathbb{Z}$, then U is also uniquely $-k$-divisible.*

Hence, it is clear that in particular we also have

Corollary 6.6. *If U is a uniquely $\mathbb{N}$-divisible, symmetric subset of a group X, then U is also uniquely $\mathbb{Z}_0$-divisible.*

Remark 6.7. By using some obvious analogues of Definition 5.1 and Remark 5.2, we can also easily see that a subset U of a vector space X over K is k-divisible (uniquely k-divisible), for some $k \in K_0$, if and only if $k^{-1}x \in U$ for all $x \in U$. That is, $k^{-1}U \subset U$.

Remark 6.8. If U is an n-cancellable subset of a groupoid X with zero, for some $n \in \mathbb{N}$, such that $0 \in U$, then $nx = 0$ implies $x = 0$ for all $x \in U$.

Namely, if $x \in U$ such that $nx = 0$, then by the corresponding definitions we also have $nx = n0$, and hence $x = 0$.

Remark 6.9. Quite similarly, we can also see that if U is a k-cancellable subset of a group X, for some $k \in \mathbb{Z}$, such that $0 \in U$, then $kx = 0$ implies $x = 0$ for all $x \in U$.

Now, by using the letter observation and Corollary 6.4, we can also easily prove

Theorem 6.10. *If U is an $\mathbb{N}$-cancellable subset of a group X such that $0 \in U$, then $kx = lx$ implies $k = l$ for all $k, l \in \mathbb{Z}$ and $x \in U_0$.*

Proof. Assume on the contrary that there exist $k, l \in \mathbb{Z}$ and $x \in U_0$ such that $kx = lx$, but $k \neq l$. Then, by using Theorem 1.8, we can see that

$$(k-l)x = \big(k+(-l)\big)x = kx + (-l)x = lx + (-l)x = \big(l+(-l)\big)x = 0x = 0.$$

Hence, by using Corollary 6.4 and Remark 6.9, we can infer that $x = 0$. This contradiction proves the theorem. $\square$

From the above theorem, by taking $l = 0$, we can immediately derive

Corollary 6.11. *If U is an $\mathbb{N}$-cancellable subset of group X such that $0 \in U$, then $kx = 0$ implies $k = 0$ for all $k \in \mathbb{Z}$ and $x \in U_0$.*

In addition to Remark 6.9, we can also easily prove the following

Theorem 6.12. *If X is a commutative group, then for each $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is k-cancellable;*

(2) *$kx = 0$ implies $x = 0$ for all $x \in X$.*

Proof. From Remark 6.9, we can see that (1) $\Longrightarrow$ (2) even if the group X is not assumed to be commutative.

Moreover, if $x, y \in X$ such that $kx = ky$, then by using Theorem 1.9 we can see that

$$k(x-y) = k(x+(-y)) = kx + k(-y) = ky + k(-y) = k(y+(-y)) = k0 = 0.$$

Hence, if (2) holds, then we can already infer that $x - y = 0$, and thus $x = y$. Therefore, (1) also holds. □

From this theorem, by using Corollary 6.4, we can immediately derive

Corollary 6.13. *If X is a commutative group such that $nx = 0$ implies $x = 0$ for all $n \in \mathbb{N}$ and $x \in X$, then X is $\mathbb{Z}_0$-cancellable.*

Remark 6.14. By using an obvious analogue of Definition 5.3, we can also easily see that every subset U of a vector space X over K is K_0-cancellable. Moreover, $kx = lx$ implies $k = l$ for all $k, \in K$ and $x \in X_0$.

7. Characterizations of divisible and cancellable sets

Definition 7.1. If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$ we define

$$n^{-1}x = \{y \in X : x = ny\}.$$

Remark 7.2. Now, having in mind the definition of the image of a set under a relation, for any $U \subset X$, we may also naturally define $n^{-1}U = \bigcup_{x \in U} n^{-1}x$.

Thus, we can easily see that $n^{-1}U = \{y \in X : ny \in U\}$. Namely, if for instance, $y \in n^{-1}U$, then by the above definition there exists $x \in U$ such that $y \in n^{-1}x$. Hence, by Definition 7.1, it already follows that $ny = x \in U$.

By using Definition 7.1, we can also easily prove the following

Theorem 7.3. *If X is a groupoid, then for any $x \in X$ and $n \in \mathbb{N}$ we have*

(1) $n(n^{-1}x) \subset \{x\}$,

(2) $\{x\} \subset n^{-1}(nx)$.

Proof. Since $nx = nx$, it is clear that $x \in n^{-1}(nx)$. Therefore, (2) is true.

Moreover, if $z \in n(n^{-1}x)$ then there exists $y \in n^{-1}x$ such that $z = ny$. Hence, since $y \in n^{-1}x$ implies $ny = x$, we can infer that $z = x$. Therefore, (1) is also true. □

Remark 7.4. Now, by using this theorem, for any $U \subset X$, we can also easily prove that $n(n^{-1}U) \subset U \subset n^{-1}(nU)$.

For instance, by using Theorem 7.3 and Remark 7.2, we can easily see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} n^{-1}(nx) = n^{-1}\Big(\bigcup_{x \in U} \{nx\}\Big) = n^{-1}(nU).$$

By using an obvious analogue of Definition 7.1, we can also easily prove the following

Theorem 7.5. *If X is a group, then for any $x \in X$ and $k \in \mathbb{Z}$ we have*

(1) $k\big(k^{-1}x\big) \subset \{x\}$,

(2) $\{x\} \subset k^{-1}\big(kx\big)$.

Remark 7.6. Now, by using this theorem, for any $U \subset X$, we can also easily prove that $k\big(k^{-1}U\big) \subset U \subset k^{-1}\big(kU\big)$.

However, it is now more important to note that, by using the corresponding definitions, we can also easily prove the following three theorems.

Theorem 7.7. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is n-divisible,*

(2) *$U \cap n^{-1}x \neq \emptyset$ for all $x \in U$.*

Theorem 7.8. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is uniquely n-divisible,*

(2) $\operatorname{card}\big(U \cap n^{-1}x\big) = 1$ *for all $x \in U$.*

Theorem 7.9. *If X is a groupoid, then for any $U \subset X$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *U is n-cancellable,*

(2) $\operatorname{card}\big(U \cap n^{-1}(nx)\big) \leq 1$ *for all $x \in U$.*

Proof. If $x \in X$ and $y_1, y_2 \in U \cap n^{-1}(nx)$, then $y_1, y_2 \in U$ and $y_1, y_2 \in n^{-1}(nx)$, and thus $ny_1 = nx = ny_2$. Hence, if (1) holds, we can infer that $y_1 = y_2$, and thus (2) also holds.

Conversely, if $x, y \in U$ such that $nx = ny$, then by Definition 7.1 we have $y \in n^{-1}(nx)$. Moreover, by Theorem 7.3, we also have $x \in n^{-1}(nx)$. Therefore, $x, y \in U \cap n^{-1}(nx)$. Hence, if (2) holds, we can infer that $x = y$. Therefore, (1) also holds. $\square$

Analogously to the latter three theorems, we can also easily prove the following three theorems.

Theorem 7.10. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is k-divisible,*

(2) *$U \cap k^{-1}x \neq \emptyset$ for all $x \in U$.*

Theorem 7.11. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is uniquely k-divisible,*

(2) $\operatorname{card}\big(U \cap k^{-1}x\big) = 1$ *for all $x \in U$.*

Theorem 7.12. *If X is a group, then for any $U \subset X$ and $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *U is k-cancellable,*

(2) $\operatorname{card}\big(U \cap k^{-1}(kx)\big) \leq 1$ *for all $x \in X$.*

Moreover, as a simple reformulation of Theorem 6.12, we can also state

Theorem 7.13. *A commutative group X, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is k-cancellable,*

(2) $k^{-1}0 \subset \{0\}$,

(3) $k^{-1}0 = \{0\}$.

Remark 7.14. Quite similarly, by Remark 6.8, we can also state that if U is an n-cancellable subset of groupoid X with zero, for some $n \in \mathbb{N}$, such that $0 \in U$, then $U \cap n^{-1}0 = \{0\}$.

Remark 7.15. Moreover, by Remark 6.9, we can also state that if U is a k-cancellable subset of group X, for some $k \in \mathbb{Z}$, such that $0 \in U$, then $U \cap k^{-1}0 = \{0\}$.

In addition to Theorem 7.13 and Remarks 7.14 and 7.15, it is also worth proving

Theorem 7.16. *The following assertions hold:*

(1) *If X is a commutative group, then $k^{-1}0$ is a subgroup of X for all $k \in \mathbb{Z}$.*

(2) *If X is a commutative monoid, then $n^{-1}0$ is a submonoid of X for all $n \in \mathbb{N}_0$.*

However, it is now more important to note that in addition to Theorems 7.7, 7.10, 7.9 and 7.12, we can also easily prove the following four theorems.

Theorem 7.17. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *X is n-divisible,*

(2) $\{x\} \subset n\big(n^{-1}x\big)$ *for all $x \in X$,*

(3) $\{x\} = n\big(n^{-1}x\big)$ *for all $x \in X$.*

Proof. If (1) holds, then by Theorem 7.7, for every $x \in X$, we have $n^{-1}x \neq \emptyset$, and thus $n\big(n^{-1}x\big) \neq \emptyset$. Moreover, by Theorem 7.3, we also have $n\big(n^{-1}x\big) \subset \{x\}$. Therefore, (3) also holds. The implication (2) $\Longrightarrow$ (1) is even more obvious by Theorem 7.7. $\square$

Theorem 7.18. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is k-divisible,*

(2) *$\{x\} \subset k\big(k^{-1}x\big)$ for all $x \in X$,*

(3) *$\{x\} = k\big(k^{-1}x\big)$ for all $x \in X$.*

Theorem 7.19. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *X is n-cancellable,*

(2) *$n^{-1}\big(nx\big) \subset \{x\}$ for all $x \in X$,*

(3) *$n^{-1}\big(nx\big) = \{x\}$ for all $x \in X$.*

Proof. If (1) holds, then by Theorem 7.9, for every $x \in X$, we have

$$\operatorname{card}\big(n^{-1}(nx)\big) \leq 1.$$

Moreover, by Theorem 7.3, we also have $\{x\} \subset n^{-1}(nx)$. Therefore, (3) also holds. The implication (2) $\Longrightarrow$ (1) is even more obvious by Theorem 7.9. □

Theorem 7.20. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is k-cancellable,*

(2) *$k^{-1}\big(kx\big) \subset \{x\}$ for all $x \in X$,*

(3) *$k^{-1}\big(kx\big) = \{x\}$ for all $x \in X$.*

Now, as some immediate consequences of the latter four theorems, and Theorems 5.6 and 5.7, we can also state the following two theorems.

Theorem 7.21. *If X is a groupoid, then for any $n \in \mathbb{N}$ the following assertions are equivalent:*

(1) *X is uniquely n-divisible,*

(2) *$n^{-1}\big(nx\big) \subset \{x\} \subset n\big(n^{-1}x\big)$ for all $x \in X$,*

(3) *$n^{-1}\big(nx\big) = \{x\} = n\big(n^{-1}x\big)$ for all $x \in X$.*

Theorem 7.22. *If X is a group, then for any $k \in \mathbb{Z}$ the following assertions are equivalent:*

(1) *X is uniquely k-divisible,*

(2) *$k^{-1}\big(kx\big) \subset \{x\} \subset k\big(k^{-1}x\big)$ for all $x \in X$,*

(3) *$k^{-1}\big(kx\big) = \{x\} = k\big(k^{-1}x\big)$ for all $x \in X$.*

8. Some further results on the sets $n^{-1}x$ and $k^{-1}x$

In addition to Theorem 7.3, we can also prove the following

Theorem 8.1. *If X is a semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have:*
(1) $m\big(n^{-1}x\big) \subset n^{-1}\big(mx\big)$,
(2) $m^{-1}\big(n^{-1}x\big) \subset (mn)^{-1}x$,
(3) $m\big((mn)^{-1}x\big) \subset n^{-1}x$,
(4) $n^{-1}x \subset (mn)^{-1}(mx)$.

Proof. If $y \in n^{-1}x$, then by Definition 7.1 we have $x = ny$. Hence, by using Theorem 1.4, we can infer that

$$mx = m(ny) = (mn)y = (nm)y = n(my).$$

Thus, by Definition 7.1, we also have

$$y \in (mn)^{-1}(mx) \qquad \text{and} \qquad my \in n^{-1}(mx).$$

Hence, we can already see that (4) and (1) are true.

On the other hand, if $y \in (mn)^{-1}x$, then by Definition 7.1 and Theorem 1.4 we have

$$x = (mn)y = (nm)y = n(my).$$

Thus, by Definition 7.1, we also have $my \in n^{-1}x$. Hence, we can already see that (3) is also true.

Finally, if $y \in m^{-1}\big(n^{-1}x\big)$, then by Remark 7.2, we have $my \in n^{-1}x$. Hence, by using Definition 7.1 and Theorem 1.4, we can infer that

$$x = n(my) = (nm)y = (mn)y.$$

Thus, by Definition 7.1, we also have $y = (mn)^{-1}x$. Hence, we can already see that (2) is also true. □

From this theorem, by Theorem 7.8, it is clear that in particular we also have

Corollary 8.2. *If X is a uniquely $\mathbb{N}$-divisible semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we have:*
(1) $m\big(n^{-1}x\big) = n^{-1}\big(mx\big)$,
(2) $m^{-1}\big(n^{-1}x\big) = (mn)^{-1}x$,
(3) $m\big((mn)^{-1}x\big) = n^{-1}x$,
(4) $n^{-1}x = (mn)^{-1}\big(mx\big)$.

Analogously to Theorem 8.1, we can also prove the following

Theorem 8.3. *If X is a group, then for any $x \in X$ and $k, l \in \mathbb{Z}$ we have:*
(1) $k\big(l^{-1}x\big) \subset l^{-1}\big(kx\big)$,
(2) $k^{-1}\big(l^{-1}x\big) \subset (kl)^{-1}x$,
(3) $k\big((kl)^{-1}x\big) \subset l^{-1}x$,
(4) $l^{-1}x \subset (kl)^{-1}\big(kx\big)$.

Hence, by Corollary 6.6 and Theorem 7.11, it is clear that in particular we have

Corollary 8.4. *If X is a uniquely $\mathbb{N}$-divisible group, then for any $x \in X$ and $k, l \in \mathbb{Z}_0$ we have:*
(1) $k\big(l^{-1}x\big) = l^{-1}\big(kx\big)$,
(2) $k^{-1}\big(l^{-1}x\big) = (kl)^{-1}x$,
(3) $k\big((kl)^{-1}x\big) = l^{-1}x$,
(4) $l^{-1}x = (kl)^{-1}\big(kx\big)$.

In addition to Theorem 8.1, we can also prove the following

Theorem 8.5. *If X is a commutative semigroup, then for any $x, y \in X$ and $n \in \mathbb{N}$ we have*
$$n^{-1}x + n^{-1}y \subset n^{-1}(x+y).$$

Proof. If $z \in n^{-1}x$ and $w \in n^{-1}y$, then by using Definition 7.1 and Theorem 1.5, we can see that
$$x + y = nz + nw = n(z+w).$$
Therefore, by Definition 7.1, we also have $z + w \in n^{-1}(x+y)$. Hence, we can already see that the required inclusion is also true. $\square$

From this theorem, by Theorem 7.8, it is clear that in particular we also have

Corollary 8.6. *If X is a uniquely $\mathbb{N}$-divisible commutative semigroup, then for any $x, y \in X$ and $n \in \mathbb{N}$ we have*
$$n^{-1}(x+y) = n^{-1}x + n^{-1}y.$$

Analogously to Theorem 8.5, we can also prove the following

Theorem 8.7. *If X is a commutative group, then for any $k \in \mathbb{Z}$ and $x, y \in X$ we have*
$$k^{-1}x + k^{-1}y \subset k^{-1}(x+y).$$

Hence, by Corollary 6.6 and Theorem 5.11, it is clear that in particular we also have

Corollary 8.8. *If X is a uniquely $\mathbb{N}$-divisible commutative semigroup, then for any $k \in \mathbb{Z}_0$ and $x, y \in X$ we have*
$$k^{-1}(x+y) = k^{-1}x + k^{-1}y.$$

Remark 8.9. In the latter two theorems and their corollaries, the commutativity assumptions on X can be weakened.

For instance, in Theorem 8.5 it would be enough to assume only that the sets $n^{-1}x$ and $n^{-1}y$ are elementwise commuting.

9. Uniquely $\mathbb{N}$-divisible semigroups

In addition to Corollary 8.2, we can also easily prove the following

Lemma 9.1. *If X is a uniquely $\mathbb{N}$-divisible semigroup and $m, n, p, q \in \mathbb{N}$ such that $m/n = p/q$, then for every $x \in X$ we have*

$$m\big(n^{-1}x\big) = p\big(q^{-1}x\big).$$

Proof. By Theorem 7.21, we have

$$n\big(n^{-1}x\big) = \{x\} = q\big(q^{-1}x\big).$$

Hence, by using that $mq = pn$, we can infer that

$$(mq)\,\big(n\big(n^{-1}x\big)\big) = (pn)\big(q\big(q^{-1}x\big)\big).$$

Now, by using Theorem 1.4, we can also see that

$$(nq)\big(m\big(n^{-1}x\big)\big) = (nq)\big(p\big(q^{-1}x\big)\big).$$

Hence, by using Theorem 5.6 and 5.11, we can see that the required equality is also true. $\square$

Analogously to this lemma, we can also prove the following

Lemma 9.2. *If X is a uniquely $\mathbb{N}$-divisible group and $n, q \in \mathbb{N}$ and $m, p \in \mathbb{Z}$ such that $m/n = p/q$, then for every $x \in X$ we have*

$$m\big(n^{-1}x\big) = p\big(q^{-1}x\big).$$

Because of the above lemmas, we may naturally introduce the following two definitions.

Definition 9.3. If X is a uniquely $\mathbb{N}$-divisible semigroup, then for any $x \in X$ and $m, n \in \mathbb{N}$ we define

$$\big(m/n\big)x = m\big(n^{-1}x\big).$$

Definition 9.4. If X is a uniquely $\mathbb{N}$-divisible group, then for any $x \in X$, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ we define

$$\big(m/n\big)x = m\big(n^{-1}x\big).$$

By using Definition 9.3 and Corollary 8.2, we can easily prove the following

Theorem 9.5. *If X is a uniquely $\mathbb{N}$-divisible semigroup, then for any $x \in X$ and $r, s \in \mathbb{Q}$, with $r, s > 0$, we have*
(1) $(r+s)x = rx + sx$,
(2) $(rs)x = r(sx)$.

Proof. By the definition of $\mathbb{Q}$, there exists $m, n, p, q \in \mathbb{N}$ such that $r = m/n$ and $s = p/q$. Now, by using Theorems 7.8 and 1.4 and Corollary 8.2, we can see that

$$\begin{aligned}(r+s)x &= \big((m/n)+(p/q)\big)x = \big((mq+pn)/(nq)\big)x \\ &= (mq+pn)\big((nq)^{-1}x\big) = (mq)\big((nq)^{-1}x\big) + (pn)\big((nq)^{-1}x\big) \\ &= m\,\big(q\big((nq)^{-1}x\big)\big) + p\,\big(n\big((nq)^{-1}x\big)\big) = m\big(n^{-1}x\big) + p\big(q^{-1}x\big) \\ &= (m/n)x + (p/q)x = rx + sx\end{aligned}$$

and

$$\begin{aligned}(rs)x &= \big((m/n)(p/q)\big)x = \big((mp)/(nq)\big)x = (mp)\big((nq)^{-1}x\big) \\ &= m\,\big(p\big((nq)^{-1}x\big)\big) = m\,\big(p\big(n^{-1}\big(q^{-1}x\big)\big)\big) = m\,\big(n^{-1}\big(p\big(q^{-1}x\big)\big)\big) \\ &= m\,\big(n^{-1}\big((p/q)x\big)\big) = (m/n)\big((p/q)x\big) = r(sx).\end{aligned}$$ □

Analogously to this theorem, we can also prove the following

Theorem 9.6. *If X is a uniquely $\mathbb{N}$-divisible group, then for any $x \in X$ and $r, s \in \mathbb{Q}$ we have*
(1) $(r+s)x = rx + sx$,
(2) $(rs)x = r(sx)$.

By using Definition 9.3 and Corollary 8.6, we can also easily prove the following

Theorem 9.7. *If X is a uniquely $\mathbb{N}$-divisible commutative semigroup, then for any $x, y \in X$ and $r \in \mathbb{Q}$, with $r > 0$, we have*

$$r(x+y) = rx + ry.$$

Proof. By the definition of $\mathbb{Q}$, there exist $m, n \in \mathbb{N}$ such that $r = m/n$. Now, by using Corollary 8.6 and Theorem 1.5, we can see that

$$\begin{aligned}r(x+y) &= (m/n)(x+y) = m\big(n^{-1}(x+y)\big) = m\big(n^{-1}x + n^{-1}y)\big) \\ &= m\big(n^{-1}x\big) + m\big(n^{-1}y\big) = m\big(n^{-1}x\big) + m\big(n^{-1}y\big) \\ &= (m/n)x + (m/n)y = rx + ry.\end{aligned}$$ □

Analogously to this theorem, we can also prove the following

Theorem 9.8. *If X is a uniquely $\mathbb{N}$-divisible commutative group, then for any $x, y \in X$ and $r \in \mathbb{Q}$, we have*

$$r(x+y) = rx + ry.$$

Now, as an immediate consequence of Theorems 9.6 and 9.7, we can also state

Corollary 9.9. *If X is a uniquely $\mathbb{N}$-divisible commutative group, then X, with the multiplication given in Definition 9.4, is a vector space over $\mathbb{Q}$.*

Remark 9.10. Note that, by Remark 6.7, every vector space X over $\mathbb{Q}$ is uniquely $\mathbb{Q}_0$-divisible.

Now, by using Corollary 9.9, from the basic decomposition theorem of vector spaces, mentioned in Remark 3.10, we can immediately derive the following

Theorem 9.11. *If X is a uniquely $\mathbb{N}$-divisible commutative group, then for each $\mathbb{N}$-divisible subgroup U of X there exists an $\mathbb{N}$-divisible subgroup V of X such that $X = U \oplus V$.*

Remark 9.12. Note that now, by Theorem 5.6, X is $\mathbb{N}$-cancellable, and thus actually both U and V are also uniquely $\mathbb{N}$-divisible. Moreover, by Corollary 6.6, U, V and X are uniquely $\mathbb{Z}_0$-divisible.

Remark 9.13. To see that the $\mathbb{N}$-divisibility of U is an essential condition in the above theorem, we can note that $\mathbb{Z}$ is an additive subgroup of the field $\mathbb{Q}$ such that, for any $\mathbb{N}$-superhomogeneous subset V of $\mathbb{Q}$ with $\mathbb{Z} \cap V \subset \{0\}$, we have $V \subset \{0\}$, and thus $\mathbb{Z} + V \subset \mathbb{Z}$.

Namely, if $x \in V$, then since $V \subset \mathbb{Q}$ there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x = m/n$. Moreover, since V is $\mathbb{N}$-superhomogeneous, we have

$$m = n(m/n) = nx \in V.$$

Hence, since $m \in \mathbb{Z}$ and $\mathbb{Z} \cap V \subset \{0\}$ also hold, we can infer that $m = 0$, and thus $x = 0$. Therefore, $V \subset \{0\}$, and thus $\mathbb{Z} + V \subset \mathbb{Z} + \{0\} = \mathbb{Z}$.

In addition to Remark 9.13, it is also worth proving the following

Theorem 9.14. *If X is an $\mathbb{N}$-cancellable group and $a \in X$, then $U = \mathbb{Z}a$ is a commutative subgroup of X such that, for every $\mathbb{N}$-divisible symmetric subset V of $X \setminus \{a\}$, we have $U \cap V \subset \{0\}$.*

Proof. By Theorems 1.8, 1.9 and 2.8, it is clear that U is a commutative subgroup of X. Therefore, we need only prove that $U \cap V \subset \{0\}$.

For this, assume on the contrary that there exists $x \in U \cap V$ such that $x \neq 0$. Then, by the definition of U, there exists $k \in Z$ such that $x = ka$. Hence, since $x \neq 0$, we can infer that $k \neq 0$. Therefore, by Corollary 6.2, there exists $v \in V$ such that $x = kv$. Thus, we have $ka = kv$. Hence, by using Corollary 6.4, we can infer that $a = v$, and thus $a \in V$. This contradiction proves the required inclusion. □

From this theorem, by using Theorem 3.4, we can immediately derive

Corollary 9.15. *If X and U are as in Theorem 9.14, then for every $\mathbb{N}$-divisible subgroup V of X with $a \notin V$ and $X = U + V$ we have $X = U \oplus V$.*

Remark 9.16. Concerning Theorem 9.11, it is also worth mentioning that Baer [1] in 1936 already proved that if U is an $\mathbb{N}$-divisible subgroup of a commutative group X, then there exists a subgroup V of X such that $X = U \oplus V$.

Moreover, Kertész [11] in 1951 proved that if X is a commutative group such that the order of each element of X is a square-free number, then for every subgroup U of X there exists a subgroup V of X such that $X = U \oplus V$.

Surprisingly, the above two results were already considered to be well-known by Baer in [1, p.1] and [3, p. 504]. Moreover, it is also worth mentioning that Hall [9], analogously to Kertész [11], also proved an "if and only if result".

10. Operations with divisible and cancellable sets

Theorem 10.1. *If U is an n-divisible subset of a semigroup X, for some $n \in \mathbb{N}$, then for every $m \in \mathbb{N}$ the set mU is also n-divisible.*

Proof. If $x \in mU$, then by the definition of mU there exists $u \in U$ such that $x = mu$. Moreover, by the n-divisibility of U, there exists $v \in U$ such that $u = nv$. Hence, by using Theorem 1.4, we can see that $x = mu = m(nv) = n(mv)$. Thus, since $mv \in mU$, the required assertion is also true. □

Moreover, as a certain converse to this theorem, we can also prove

Theorem 10.2. *If U is an m-cancellable, n-superhomogeneous subset of a semigroup X, for some $m, n \in \mathbb{N}$, such that mU is n-divisible, then U is also n-divisible.*

Proof. If $x \in U$, then by the definition mU we also have $mx \in mU$. Therefore, by the n-divisibility of mU, there exists $v \in mU$ such that $mx = nv$. Moreover, by the definition of mU, there exists $y \in U$ such that $v = my$. Now, by using Theorem 1.4, we can see that $mx = nv = n(my) = m(ny)$. Hence, by using the m-cancellability of U and the fact that $ny \in U$, we can already infer that $x = ny$. Therefore, the required assertion is also true. □

Quite similarly to Theorems 10.1 and 10.2, we can also prove the following two theorems.

Theorem 10.3. *If U is a k-divisible subset of a group X, for some $k \in \mathbb{Z}$, then for every $l \in \mathbb{Z}$ the set lU is also k-divisible.*

Theorem 10.4. *If U is an l-cancellable, k-superhomogeneous subset of a group X, for some $l, k \in \mathbb{N}$, such that lU is k-divisible, then U is also k-divisible.*

In addition to Theorem 10.1, we can also easily prove the following

Theorem 10.5. *If U and V are elementwise commuting, n-divisible subsets of a semigroup X, for some $n \in \mathbb{N}$, then $U + V$ is also n-divisible.*

Proof. If $x \in U + V$, then by the definition of $U + V$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Moreover, since U and V are n-divisible, there exist $\omega \in U$ and $w \in V$ such that $u = n\omega$ and $v = nw$. Hence, by using Theorem 1.5, we can see that $x = u + v = n\omega + nw = n(\omega + w)$. Thus, since $\omega + w \in U + V$, the required assertion is also true. $\square$

Moreover, as a certain converse to this theorem, we can also prove

Theorem 10.6. *If U and V are elementwise commuting, n-superhomogeneous subsets of a monoid X, for some $n \in \mathbb{N}$, such that $U + V$ is n-divisible, and $U + V = U \oplus V$ and $0 \in V$, then U is also n-divisible.*

Proof. If $x \in U$, then because of $0 \in V$ we also have $x \in U + V$. Thus, by the n-divisibility of $U + V$, there exists $y \in U + V$ such that $x = ny$. Moreover, by the definition of $U + V$, there exist $u \in U$ and $v \in V$ such that $y = u + v$. Now, by using Theorem 1.5, we can see that

$$x = ny = n(u + v) = nu + nv.$$

Moreover, we can also note that $x \in U + V$, $nu \in U$ and $nv \in V$. Hence, since $x = x + 0$ also holds with $x \in U$ and $0 \in V$, by using the assumption $U + V = U \oplus V$, we can already infer that $x = nu$. Therefore, U is also n-divisible. $\square$

Quite similarly to Theorems 10.5 and 10.6, we can also prove the following two theorems.

Theorem 10.7. *If U and V are elementwise commuting, k-divisible subsets of a semigroup X, for some $k \in \mathbb{Z}$, then $U + V$ is also k-divisible.*

Theorem 10.8. *If U and V are elementwise commuting, k-superhomogeneous subsets of a group X, for some $k \in \mathbb{Z}$, such that $U + V$ is k-divisible, and $U + V = U \oplus V$ and $0 \in V$, then U is also k-divisible.*

Hence, by Theorem 3.4, it is clear that in particular we also have

Corollary 10.9. *If U and V are elementwise commuting subgroups of a group X such that $U + V$ is k-divisible, for some $k \in \mathbb{Z}$ such that $U \cap V = \{0\}$, then U and V are also n-divisible.*

In addition to Theorem 10.5, we can also prove the following

Theorem 10.10. *If U and V are elementwise commuting, n-superhomogeneous subsets of a semigroup X, for some $n \in \mathbb{N}$ such that U and V are n-cancellable and $U + V = U \oplus V$, then $U + V$ is also n-cancellable.*

Proof. For this, assume that $x, y \in U + V$ such $nx = ny$. Then, by the definition of $U + V$, there exist $u, \omega \in U$ and $v, w \in V$ such that $x = u + v$ and $y = \omega + w$. Hence, by using Theorem 1.5, we can see that

$$nu + nv = n(u + v) = nx = ny = n(\omega + w) = n\omega + nw.$$

Moreover, we can also note that $nu, n\omega \in U$ and $nv, nw \in V$, and thus $nu + nv, n\omega + nw \in U + V$. Now, by using that $U + V = U \oplus V$, we can see that $nu = n\omega$ and $nv = nw$. Hence, by using the n-cancellability of U and V, we can already infer that $u = \omega$ and $v = w$. Therefore, $x = u + v = \omega + w = y$, and thus the required assertion is also true. □

Remark 10.11. Now, as a trivial converse to this theorem, we can also state that if U and V subsets of a monoid X such that $U + V$ is n-cancellable, for some $n \in \mathbb{Z}$, and $0 \in U \cap V$, then U and V are also n-cancellable.

Quite similarly to Theorem 10.10, we can also prove the following

Theorem 10.12. *If U and V are elementwise commuting, k-superhomogeneous subsets of a group X, for some $k \in \mathbb{Z}$ such that U and V are k-cancellable and $U + V = U \oplus V$, then $U + V$ is also k-cancellable.*

Hence, by Theorem 3.4, it is clear that in particular we also have

Corollary 10.13. *If U and V are elementwise commuting subgroups of a group X such that U and V are k-cancellable for some $k \in \mathbb{Z}$, and $U \cap V = \{0\}$, then $U + V$ is also k-cancellable.*

Remark 10.14. In an immediate continuation of this paper, by using the notion of the order

$$n_a = \inf\{n \in \mathbb{N} : na = 0\}$$

of an element a of a monoid (resp. group) X, we shall investigate the divisibility and cancellability properties of the set $\mathbb{N}_0 a + V$ (resp. $\mathbb{Z}a + V$) for some substructures V of X.

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