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Algebraic relations with the infinite products generated by Fibonacci numbers

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Abstract

In this paper, we establish explicit algebraic relations among infinite products including Fibonacci and Lucas numbers with subscripts in geometric progressions. The algebraic relations given in this paper are obtained by using general criteria for the algebraic dependency of such infinite products.

Keywords: Algebraic independence, Infinite products, Fibonacci numbers, Mahler functions.

MSC: 11J81, 11J85.

1. Introduction

Let α and β be real algebraic numbers with $|\alpha| > 1$ and $\alpha\beta = -1$. We define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad (n \geq 0). \tag{1.1}$$

If $\alpha = (1+\sqrt{5})/2$, we have $U_n = F_n$ and $V_n = L_n$ $(n \geq 0)$, where the sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are the Fibonacci numbers and the Lucas numbers defined, respectively, by $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and by $L_{n+2} = L_{n+1} + L_n$ $(n \geq 0)$, $L_0 = 2$, $L_1 = 1$.

Throughout this paper, we adopt the following notation. Let $d \geq 2$ be a fixed integer and $\zeta_m = e^{2\pi i/m}$ a primitive m-th root of unity. For $\tau \in \mathbb{C}$ with $|\tau| = 1$, we define the set $\Omega_j(\tau) := \left\{ z \in \mathbb{C} \mid z^{d^j} = \tau \text{ or } z^{d^j} = \overline{\tau} \right\}$ for $j = 0, 1, \ldots$. Let $S_k(\tau)$ be a subset of $\Omega_k(\tau)$ such that for any $\gamma \in S_k(\tau)$ the numbers $\zeta_d \gamma$ and $\overline{\gamma}$ belong to $S_k(\tau)$, where $\overline{\gamma}$ indicates the complex conjugate of γ. Namely, $S_k(\tau)$ satisfies $S_k(\tau) = \zeta_d S_k(\tau)$ and $S_k(\tau) = \overline{S_k(\tau)}$. For example, if $d = 2$, $\tau = 1$, and $k = 3$ we have $\Omega_3(1) = \{e^{k\pi i/4} \mid 0 \leq k \leq 7\}$ and so we can choose $S_3(1) = \{\pm e^{\pi i/4}, \pm e^{3\pi i/4}\}$. We define the following sets that are determined depending only on $S_k(\tau)$:

$$\Lambda_i(\tau) = \left\{ \gamma^{d^{k-i}} \mid \gamma \in S_k(\tau) \right\} \quad (0 \leq i \leq k-1),$$

$$\Gamma_i(\tau) = \{\gamma \in \Omega_i(\tau) \mid \gamma^d \in \Lambda_{i-1}(\tau)\} \setminus \Lambda_i(\tau) \quad (1 \leq i \leq k-1).$$

Then we put

$$\mathcal{E}_k(\tau) = \left(\bigcup_{i=1}^{k-1} \Gamma_i(\tau) \right) \bigcup S_k(\tau) \tag{1.2}$$

and

$$\mathcal{F}_k(\tau) = \begin{cases} \mathcal{E}_k(\tau) \bigcup \{\tau, \overline{\tau}\} & \text{if } \tau \notin \mathcal{E}_k(\tau), \\ \mathcal{E}_k(\tau) \setminus \{\tau, \overline{\tau}\} & \text{otherwise.} \end{cases}$$

In [1] we established necessary and sufficient conditions for the infinite products generated by each of the sequences in (1.1) to be algebraically dependent over $\mathbb{Q}$ and obtained the following:

Theorem 1.1. *Let $\{U_n\}_{n\geq 0}$ be the sequence defined by* (1.1) *and d be an integer greater than* 1*. Let $a_1, \ldots, a_m$ be nonzero distinct real algebraic numbers. Then the numbers*

$$\prod_{\substack{k=0 \\ U_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{U_{d^k}} \right) \quad (i = 1, \ldots, m)$$

are algebraically dependent if and only if d is odd and there exist distinct τ_1, $\tau_2 \in \mathbb{C}$ with $|\tau_1| = |\tau_2| = 1$ and $\mathcal{F}_{k_1}(\tau_1), \mathcal{F}_{k_2}(\tau_2)$ for some $k_1, k_2 \geq 1$ such that $\mathcal{F}_{k_1}(\tau_1) \cap \mathcal{F}_{k_2}(\tau_2) \subset \{\tau_1, \overline{\tau_1}, \tau_2, \overline{\tau_2}\}$ and $\{a_1, \ldots, a_m\}$ contains

$$-\frac{1}{\alpha - \beta}(\gamma + \overline{\gamma})$$

for all $\gamma \in (\mathcal{F}_{k_1}(\tau_1) \bigcup \mathcal{F}_{k_2}(\tau_2)) \setminus \{\pm\sqrt{-1}\}$.

Theorem 1.2. *Let $\{V_n\}_{n\geq 0}$ be the sequence defined by* (1.1) *and d be an integer greater than* 1*. Let $a_1, \ldots, a_m$ be nonzero distinct real algebraic numbers. Then the numbers*

$$\prod_{\substack{k=0 \\ V_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{V_{d^k}} \right) \quad (i = 1, \ldots, m)$$

are algebraically dependent if and only if at least one of the following properties is satisfied:

1. *$d = 2$ and the set $\{a_1, \dots, a_m\}$ contains $b_1, \dots, b_l$ $(l \geq 3)$ with $b_1 < -2$ satisfying*
$$b_2 = -b_1, \quad b_j = b_{j-1}^2 - 2 \quad (j = 3, \dots, l-1), \quad b_l = -b_{l-1}^2 + 2.$$
2. *$d = 2$ and there exist $\tau \in \mathbb{C}$ with $|\tau| = 1$ and $\mathcal{F}_k(\tau)$ for some $k \geq 1$ such that $\{a_1, \dots, a_m\}$ contains*
$$-(\gamma + \overline{\gamma})$$
for all $\gamma \in \mathcal{F}_k(\tau) \setminus \{\pm\sqrt{-1}\}$.
3. *$d \geq 4$ is even and there exist distinct τ_1, $\tau_2 \in \mathbb{C}$ with $|\tau_1| = |\tau_2| = 1$ and $\mathcal{F}_{k_1}(\tau_1), \mathcal{F}_{k_2}(\tau_2)$ for some $k_1, k_2 \geq 1$ such that $\mathcal{F}_{k_1}(\tau_1) \cap \mathcal{F}_{k_2}(\tau_2) \subset \{\tau_1, \overline{\tau_1}, \tau_2, \overline{\tau_2}\}$ and $\{a_1, \dots, a_m\}$ contains*
$$-(\gamma + \overline{\gamma})$$
for all $\gamma \in (\mathcal{F}_{k_1}(\tau_1) \bigcup \mathcal{F}_{k_2}(\tau_2)) \setminus \{\pm\sqrt{-1}\}$.

Note that Theorems 1.1 and 1.2 above are generalizations of [2, Theorems 1 and 2], respectively.

Corollary 1.3 (cf. [3])**.** *Let $d \geq 2$ be a fixed integer and $a \neq 0$ a real algebraic number. Then the numbers*
$$\prod_{\substack{k=1 \\ U_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{U_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1 \\ V_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{V_{d^k}}\right)$$
are transcendental, except for only two algebraic numbers
$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{V_{2^k}}\right) = \frac{\alpha^4 - 1}{\alpha^4 + \alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{V_{2^k}}\right) = \frac{\alpha^2 + 1}{\alpha^2 - 1}. \tag{1.3}$$

Corollary 1.4. *Let a be a nonzero real algebraic number with $a \neq -V_{2^k} - 2$ $(k \geq 1)$. Then the number*
$$\prod_{k=1}^{\infty} \left(1 + \frac{a}{V_{2^k} + 2}\right)$$
is transcendental, except when $a = -3, -2$; indeed
$$\prod_{k=1}^{\infty} \left(1 - \frac{2}{V_{2^k} + 2}\right) = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 - \frac{3}{V_{2^k} + 2}\right) = \frac{(\alpha^2 - 1)^2}{\alpha^4 + \alpha^2 + 1}. \tag{1.4}$$

Proof. Using the equality

$$1+\frac{a}{V_{2^k}+2}=\left(1+\frac{a+2}{V_{2^k}}\right)\left(1+\frac{2}{V_{2^k}}\right)^{-1}$$

and the second equality in (1.3), we have

$$\prod_{k=1}^{\infty}\left(1+\frac{a}{V_{2^k}+2}\right)=\frac{\alpha^2-1}{\alpha^2+1}\prod_{k=1}^{\infty}\left(1+\frac{a+2}{V_{2^k}}\right). \tag{1.5}$$

By Corollary 1.3 we see that the infinite product in the right-hand side of (1.5) is algebraic only if $a=-3,-2$. The equalities (1.4) follow immediately from (1.5) with (1.3). □

Applying Corollary 1.4 with $\alpha=(1+\sqrt{5})/2$, we obtain the transcendence of

$$\prod_{k=1}^{\infty}\left(1+\frac{a}{L_{2^k}+2}\right)$$

for any nonzero algebraic number $a\neq-3,-2,-L_{2^k}-2$ $(k\geq 1)$, and the equalities

$$\prod_{k=1}^{\infty}\left(1-\frac{2}{L_{2^k}+2}\right)=\frac{1}{\sqrt{5}},\quad \prod_{k=1}^{\infty}\left(1-\frac{3}{L_{2^k}+2}\right)=\frac{1}{4}. \tag{1.6}$$

It should be noted that Corollaries 1.3 and 1.4 hold even if the number a is a nonzero complex algebraic number (see [3]).

2. Algebraic dependence relations

Theorems 1.1 and 1.2 in the introduction are useful to obtain the explicit algebraic dependence relations among the infinite products generated by the Fibonacci and Lucas numbers as well as their transcendence degrees. We exhibit such examples in this section and their proofs in the next section.

Example 2.1. Let a be a nonzero real algebraic number. The transcendental numbers

$$s_1=\prod_{\substack{k=0\\F_{3^k}\neq-a}}^{\infty}\left(1+\frac{a}{F_{3^k}}\right),\quad s_2=\prod_{\substack{k=0\\F_{3^k}\neq a}}^{\infty}\left(1-\frac{a}{F_{3^k}}\right)$$

are algebraically dependent if and only if $a=\pm 1/\sqrt{5}$. If $a=1/\sqrt{5}$, then

$$s_1 s_2^{-1}=2+\sqrt{5}.$$

Example 2.2. The transcendental numbers

$$s_1 = \prod_{k=0}^{\infty}\left(1+\frac{a_1}{F_{5^k}}\right),\ s_2 = \prod_{k=0}^{\infty}\left(1+\frac{a_2}{F_{5^k}}\right),$$

$$s_3 = \prod_{k=0}^{\infty}\left(1-\frac{a_1}{F_{5^k}}\right),\ s_4 = \prod_{k=0}^{\infty}\left(1-\frac{a_2}{F_{5^k}}\right)$$

with $a_1 = (-5+\sqrt{5})/10,\ a_2 = (5+\sqrt{5})/10$ satisfy

$$s_1 s_2 s_3^{-1} s_4^{-1} = 2+\sqrt{5},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4) = 3$.

Remark 2.3. The infinite products $\prod_{k=0}^{\infty}(1+a_i/F_{d^k})$ for odd d and $\prod_{k=1}^{\infty}(1+a_i/L_{d^k})$ for even d are easily expressed as the values at an algebraic number of $\Phi_i(z)$ defined by (3.2) with $b=1$, which will be shown in (3.3) of Section 3. Hence, for simplicity, we take $k \geq 1$ in the following examples.

Example 2.4. Let $a \neq 2, -1$ be a real algebraic number. The transcendental numbers

$$s_1 = \prod_{\substack{k=1 \\ L_{2^k} \neq -a}}^{\infty}\left(1+\frac{a}{L_{2^k}}\right), \quad s_2 = \prod_{\substack{k=1 \\ L_{2^k} \neq a}}^{\infty}\left(1-\frac{a}{L_{2^k}}\right)$$

are algebraically dependent if and only if $a = \pm\sqrt{2}$. If $a = \pm\sqrt{2}$, using the relation $L_{2^k}^2 = L_{2^{k+1}} + 2$ $(k \geq 1)$ and the first equality in (1.6), we have

$$s_1 s_2 = \prod_{k=2}^{\infty}\left(1-\frac{2}{L_{2^k}+2}\right) = \frac{5}{3}\cdot\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{3}.$$

Example 2.5. The transcendental numbers

$$s_1 = \prod_{k=1}^{\infty}\left(1-\frac{\sqrt{3}}{L_{4^k}}\right), s_2 = \prod_{k=1}^{\infty}\left(1+\frac{\sqrt{3}}{L_{4^k}}\right),$$

$$s_3 = \prod_{k=1}^{\infty}\left(1-\frac{1}{L_{4^k}}\right), s_4 = \prod_{k=1}^{\infty}\left(1+\frac{2}{L_{4^k}}\right)$$

satisfy

$$s_1 s_2 s_3 s_4^{-1} = \frac{5}{8},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4) = 3$.

Example 2.6. The transcendental numbers

$$s_1 = \prod_{k=1}^{\infty}\left(1-\frac{1}{L_{6^k}}\right), \quad s_2 = \prod_{k=1}^{\infty}\left(1+\frac{1}{L_{6^k}}\right), \quad s_3 = \prod_{k=1}^{\infty}\left(1+\frac{2}{L_{6^k}}\right),$$

$$s_4 = \prod_{k=1}^{\infty} \left(1 + \frac{\sqrt{3}}{L_{6^k}}\right), \quad s_5 = \prod_{k=1}^{\infty} \left(1 - \frac{\sqrt{3}}{L_{6^k}}\right)$$

satisfy

$$s_1 s_2 s_3 s_4^{-1} s_5^{-1} = \frac{\sqrt{5}}{2},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4, s_5) = 4$.

Example 2.7. The transcendental numbers

$$s_i = \prod_{k=1}^{\infty} \left(1 + \frac{a_i}{L_{4^k}}\right) \quad (i = 1, \ldots, 8),$$

where

$$a_1 = -(\zeta_{16}^1 + \zeta_{16}^{15}), \quad a_2 = -(\zeta_{16}^5 + \zeta_{16}^{11}), \quad a_3 = -(\zeta_{16}^7 + \zeta_{16}^9), \quad a_4 = -(\zeta_{64}^3 + \zeta_{64}^{61}),$$

$$a_5 = -(\zeta_{64}^{13} + \zeta_{64}^{51}), \quad a_6 = -(\zeta_{64}^{19} + \zeta_{64}^{45}), \quad a_7 = -(\zeta_{64}^{29} + \zeta_{64}^{35}), \quad a_8 = 2,$$

satisfy

$$s_1 s_2 \cdots s_7 s_8^{-2} = \frac{25}{7(7 - \sqrt{2 - \sqrt{2}})}.$$

Example 2.8. The transcendental numbers

$$s_i = \prod_{k=1}^{\infty} \left(1 + \frac{a_i}{L_{4^k}}\right) \quad (i = 1, \ldots, 10),$$

where

$$a_1 = -\frac{3}{2}, \quad a_2 = \frac{\sqrt{7}}{2}, \quad a_3 = \frac{3}{2}, \quad a_4 = -\frac{\sqrt{7}}{2}, \quad a_5 = \frac{31}{16},$$

$$a_6 = -\frac{4}{\sqrt{5}}, \quad a_7 = \frac{2}{\sqrt{5}}, \quad a_8 = \frac{4}{\sqrt{5}}, \quad a_9 = -\frac{2}{\sqrt{5}}, \quad a_{10} = \frac{14}{25},$$

satisfy

$$s_1 s_2 s_3 s_4 s_5^{-1} s_6^{-1} s_7^{-1} s_8^{-1} s_9^{-1} s_{10} = \frac{3024}{3575},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, \ldots, s_{10}) = 9$.

3. Proofs of the examples

Let $\{R_n\}_{n\geq 0}$ be the sequence $\{U_n\}_{n\geq 0}$ or $\{V_n\}_{n\geq 0}$ defined by (1.1). Let $d \geq 2$ be a fixed integer and $a_1, \ldots, a_m$ nonzero real algebraic numbers. Define

$$(p_i, b) := \begin{cases} ((\alpha - \beta)a_i, -(-1)^d) & \textit{if } R_n = U_n, \\ (a_i, (-1)^d) & \textit{if } R_n = V_n, \end{cases} \tag{3.1}$$

and

$$\Phi_i(z) := \prod_{k=0}^{\infty} \left(1 + \frac{p_i z^{d^k}}{1 + bz^{2d^k}}\right) \quad (i = 1, \ldots, m). \tag{3.2}$$

Taking an integer $N \geq 1$ such that $|R_{d^k}| > \max\{|a_1|, \ldots, |a_m|\}$ for all $k \geq N$, we have

$$\begin{aligned}\Phi_i(\alpha^{-d^N}) &= \prod_{k=N}^{\infty} \left(1 + \frac{p_i \alpha^{-d^k}}{1 + b\alpha^{-2d^k}}\right) \\ &= \prod_{k=N}^{\infty} \left(1 + \frac{p_i}{\alpha^{d^k} + b(-1)^{d^k} \beta^{d^k}}\right) = \prod_{k=N}^{\infty} \left(1 + \frac{a_i}{R_{d^k}}\right) \quad (i = 1, \ldots, m),\end{aligned}$$

so that

$$\prod_{\substack{k=0 \\ R_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{R_{d^k}}\right) = \Phi_i(\alpha^{-d^N}) \prod_{\substack{k=0 \\ R_{d^k} \neq -a_i}}^{N-1} \left(1 + \frac{a_i}{R_{d^k}}\right) \quad (i = 1, \ldots, m). \tag{3.3}$$

We note that (3.3) is valid also for $N = 0$ only if d is odd and $R_{d^k} \neq -a_i$ $(k \geq 0)$.

Proof of Example 2.1. First we show that s_1 and s_2 are algebraically dependent only if $a = \pm 1/\sqrt{5}$, using the case of $m = 2$ in Theorem 1.1. If s_1 and s_2 are algebraically dependent, then $\{\tau_1, \tau_2\} = \{1, -1\}$, since $\mathcal{F}_k(\tau)$ consists of at least four elements if $\tau \neq \pm 1$. If $d = 3$, $m = 2$, and $\{\tau_1, \tau_2\} = \{1, -1\}$, it is easily seen that $\mathcal{F}_1(\tau_1) \bigcup \mathcal{F}_1(\tau_2) = \{\zeta_3, \overline{\zeta_3}, -\zeta_3, -\overline{\zeta_3}\}$ and so $\{a_1, a_2\} = \{1/\sqrt{5}, -1/\sqrt{5}\}$.

Next we show the equality $s_1 s_2^{-1} = 2 + \sqrt{5}$ by proving a general relation which holds for the functions $\Phi_i(z)$ $(1 \leq i \leq d-1)$ defined by (3.2), where $d \geq 3$ is an odd integer. Put

$$p_1 = -(\zeta_d + \overline{\zeta_d}),\ p_2 = -(\zeta_d^2 + \overline{\zeta_d}^2), \ldots,\ p_{\frac{d-1}{2}} = -(\zeta_d^{\frac{d-1}{2}} + \overline{\zeta_d}^{\frac{d-1}{2}})$$

in the equation (3.2) with $b = 1$. Then we have

$$\begin{aligned}&\Phi_1(z) \cdots \Phi_{\frac{d-1}{2}}(z) \\ &= \prod_{k=0}^{\infty} \left(\frac{1}{(1 + z^{2d^k})^{\frac{d-1}{2}}} \frac{1 - z^{d^{k+1}}}{1 - z^{d^k}}\right) = \frac{1}{1 - z} \prod_{k=0}^{\infty} \frac{1}{(1 + z^{2d^k})^{\frac{d-1}{2}}}.\end{aligned}$$

Moreover, putting

$$p_{\frac{d-1}{2}+1} = \zeta_d + \overline{\zeta_d},\ p_{\frac{d-1}{2}+2} = \zeta_d^2 + \overline{\zeta_d}^2, \ldots,\ p_{d-1} = \zeta_d^{\frac{d-1}{2}} + \overline{\zeta_d}^{\frac{d-1}{2}},$$

we get

$$\Phi_{\frac{d-1}{2}+1}(z) \cdots \Phi_{d-1}(z)$$

$$= \prod_{k=0}^{\infty} \left(\frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}} \frac{1+z^{d^{k+1}}}{1+z^{d^k}} \right) = \frac{1}{1+z} \prod_{k=0}^{\infty} \frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}}.$$

Hence, we have

$$\Phi(z) := \frac{\Phi_1(z) \cdots \Phi_{\frac{d-1}{2}}(z)}{\Phi_{\frac{d-1}{2}+1}(z) \cdots \Phi_{d-1}(z)} = \frac{1+z}{1-z}. \tag{3.4}$$

If $d=3$, then $p_1 = -(\zeta_3 + \overline{\zeta_3}) = 1$, $p_2 = \zeta_3 + \overline{\zeta_3} = -1$, and so

$$a_1 = \frac{1}{\alpha - \beta} p_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = \frac{1}{\alpha - \beta} p_2 = -\frac{1}{\sqrt{5}}$$

by (3.1). Then, by the equation (3.3) with $N=0$, we have

$$\Phi(\alpha^{-1}) = s_1 {s_2}^{-1} = \frac{\alpha+1}{\alpha-1} = 2\alpha + 1 = 2 + \sqrt{5}. \qquad \square$$

Proof of Example 2.2. We consider the case of $d=5$ in (3.4). Then

$$p_1 = -(\zeta_5 + \overline{\zeta_5}) = \frac{1-\sqrt{5}}{2}, \quad p_2 = -(\zeta_5^2 + \overline{\zeta_5^2}) = \frac{1+\sqrt{5}}{2},$$

$$p_3 = \zeta_5 + \overline{\zeta_5} = \frac{-1+\sqrt{5}}{2}, \quad p_4 = \zeta_5^2 + \overline{\zeta_5^2} = -\frac{1+\sqrt{5}}{2}.$$

By (3.1) we have

$$a_1 = \frac{-5+\sqrt{5}}{10}, \qquad a_2 = \frac{5+\sqrt{5}}{10}, \qquad a_3 = \frac{5-\sqrt{5}}{10}, \qquad a_4 = -\frac{5+\sqrt{5}}{10}.$$

Then, by the equation (3.3) with $N=0$ and (3.4), we have

$$\Phi(\alpha^{-1}) = \frac{s_1 s_2}{s_3 s_4} = \frac{\alpha+1}{\alpha-1} = 2 + \sqrt{5}.$$

Finally, we prove that $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4) = 3$, using Theorem 1.1. Let $\tau_1 = 1$, $\tau_2 = -1$, $S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2}, 1\}$, and $S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{-\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2}, -1\}$. Then $\mathcal{F}_1(\tau_1) = \{\zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2}\}$ and $\mathcal{F}_1(\tau_2) = \{-\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2}\}$. It is enough to show that s_1, s_2, and s_3 are algebraically independent, which is equivalent to the fact that a_1, a_2, and a_3 do not satisfy Theorem 1.1 with $m=3$. By (1.2) with $S_1(\tau_i) = \mathcal{E}_1(\tau_i)$ $(i=1,2)$, considering the number of the elements of $S_k(\tau_i)$ with $k \geq 2$ satisfying $S_k(\tau_i) = \zeta_5 S_k(\tau_i)$ and $S_k(\tau_i) = \overline{S_k(\tau_i)}$, we see that $\{a_1, a_2, a_3, a_4\}$ is the minimal set of $-(\gamma + \overline{\gamma})/\sqrt{5}$ with $\gamma \in \mathcal{F}_{k_1}(\tau_1) \cup \mathcal{F}_{k_2}(\tau_2) \setminus \{\pm\sqrt{-1}\}$ satisfying Theorem 1.1 with $m=4$. $\square$

Proof of Example 2.4. First we prove directly that $s_1 s_2 = \sqrt{5}/3$ if $a = \pm\sqrt{2}$. Let $\tau = \sqrt{-1}$ and $S_1(\tau) = \mathcal{E}_1(\tau) = \{\zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8}\}$ in the property 2 of Theorem 1.2. Then $\mathcal{F}_1(\tau) = \{\zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8}, \sqrt{-1}, -\sqrt{-1}\}$. Putting

$$p_1 = -(\zeta_8 + \overline{\zeta_8}) = -\sqrt{2}, \quad p_2 = \zeta_8 + \overline{\zeta_8} = \sqrt{2}$$

in the equation (3.2) with $b = 1$, we have

$$\begin{aligned}\Phi_1(z)\Phi_2(z) &= \prod_{k=0}^{\infty}(z^{2^k} - \zeta_8)(z^{2^k} - \overline{\zeta_8})(z^{2^k} + \zeta_8)(z^{2^k} + \overline{\zeta_8})\frac{1}{(1+z^{2\cdot 2^k})^2} \\ &= \prod_{k=0}^{\infty}\frac{1+z^{2^{k+2}}}{(1+z^{2^{k+1}})^2} = \prod_{k=0}^{\infty}\frac{1+z^{2^{k+2}}}{1+z^{2^{k+1}}}\frac{1-z^{2^{k+1}}}{1-z^{2^{k+2}}} = \frac{1-z^2}{1+z^2}.\end{aligned}$$

By the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$, we get

$$s_1 s_2 = \Phi_1(\alpha^{-2})\Phi_2(\alpha^{-2}) = \frac{\alpha^4 - 1}{\alpha^4 + 1}.$$

Hence, noting that $\alpha^4 = (\alpha+1)^2 = 3\alpha + 2$, we have

$$s_1 s_2 = \frac{1}{3}\cdot\frac{3\alpha+1}{\alpha+1} = \frac{1}{3}(2\alpha - 1) = \frac{\sqrt{5}}{3}.$$

Conversely, if s_1 and s_2 are algebraically dependent for some algebraic number a, then by the property 2 of Theorem 1.2 with $m = 2$ the set $\mathcal{F}_k(\tau) \setminus \{\pm\sqrt{-1}\}$ must consist of four elements, which is achieved only if $\tau = \pm\sqrt{-1}$ and $k = 1$. □

Proof of Example 2.5. We use the property 3 of Theorem 1.2. Let $\tau_1 = \zeta_3$, $\tau_2 = 1$, $S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_{12}, \overline{\zeta_{12}}, \zeta_{12}^4, \overline{\zeta_{12}^4}, \zeta_{12}^5, \overline{\zeta_{12}^5}, \zeta_{12}^2, \overline{\zeta_{12}^2}\}$, and $S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. Then $\mathcal{F}_1(\tau_1) = \{\zeta_{12}, \overline{\zeta_{12}}, \zeta_{12}^5, \overline{\zeta_{12}^5}, \zeta_{12}^2, \overline{\zeta_{12}^2}\}$ and $\mathcal{F}_1(\tau_2) = \{-1, \sqrt{-1}, -\sqrt{-1}\}$. Putting

$$p_1 = -(\zeta_{12} + \overline{\zeta_{12}}) = -\sqrt{3}, \quad p_2 = -(\zeta_{12}^5 + \overline{\zeta_{12}^5}) = \sqrt{3}, \quad p_3 = -(\zeta_{12}^2 + \overline{\zeta_{12}^2}) = -1,$$

and $p_4 = 2$ in the equation (3.2) with $b = 1$, we have

$$\begin{aligned}&\Phi_1(z)\Phi_2(z)\Phi_3(z) \\ &= \prod_{k=0}^{\infty}(z^{4^k} - \zeta_{12})(z^{4^k} - \overline{\zeta_{12}})(z^{4^k} - \zeta_{12}^5)(z^{4^k} - \overline{\zeta_{12}^5})(z^{4^k} - \zeta_{12}^2)(z^{4^k} - \overline{\zeta_{12}^2})\frac{1}{(1+z^{2\cdot 4^k})^3} \\ &= \prod_{k=0}^{\infty}\frac{(z^{4^{k+1}} - \zeta_{12}^4)(z^{4^{k+1}} - \overline{\zeta_{12}^4})}{(z^{4^k} - \zeta_{12}^4)(z^{4^k} - \overline{\zeta_{12}^4})}\frac{1}{(1+z^{2\cdot 4^k})^3} \\ &= \frac{1}{(z - \zeta_{12}^4)(z - \overline{\zeta_{12}^4})}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3},\end{aligned}$$

and

$$\begin{aligned}\Phi_4(z) &= \prod_{k=0}^{\infty}\frac{1+2z^{4^k}+z^{2\cdot 4^k}}{1+z^{2\cdot 4^k}} = \prod_{k=0}^{\infty}\frac{(1+z^{4^k})^2(1+z^{2\cdot 4^k})^2}{(1+z^{2\cdot 4^k})^3} \\ &= \prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3}\left(\frac{1-z^{4^{k+1}}}{1-z^{4^k}}\right)^2 = \frac{1}{(1-z)^2}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3}. \qquad (3.5)\end{aligned}$$

Hence, we get

$$\Phi(z) := \Phi_1(z)\Phi_2(z)\Phi_3(z)\Phi_4^{-1}(z) = \frac{(1-z)^2}{1+z+z^2}.$$

By the equation (3.3) with $N=1$ and $\alpha=(1+\sqrt{5})/2$, we have

$$s_1s_2s_3s_4^{-1} = \Phi(\alpha^{-4}) = \frac{\alpha^8-2\alpha^4+1}{\alpha^8+\alpha^4+1} = \frac{7\alpha^4-2\alpha^4}{7\alpha^4+\alpha^4} = \frac{5}{8},$$

since

$$\alpha^8+1 = (3\alpha+2)^2+1 = 21\alpha+14 = 7\alpha^4. \tag{3.6}$$

To prove that $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1,s_2,s_3,s_4)=3$, it is enough to show that s_2, s_3, and s_4 are algebraically independent, which is equivalent to the fact that p_2, p_3, and p_4 do not satisfy the property 3 of Theorem 1.2 with $m=3$. By (1.2) with $S_1(\tau_i)=\mathcal{E}_1(\tau_i)$ $(i=1,2)$, considering the number of the elements of $S_k(\tau_i)$ with $k\geq 2$ satisfying $S_k(\tau_i)=\zeta_4 S_k(\tau_i)$ and $S_k(\tau_i)=\overline{S_k(\tau_i)}$, we see that $\{-\sqrt{3},\sqrt{3},-1,2\}$ is the minimal set of $-(\gamma+\overline{\gamma})$ with $\gamma\in\mathcal{F}_{k_1}(\tau_1)\cup\mathcal{F}_{k_2}(\tau_2)\setminus\{\pm\sqrt{-1}\}$ satisfying the property 3 of Theorem 1.2 with $m=4$. $\square$

Proof of Example 2.6. We use the property 3 of Theorem 1.2. Let $\tau_1=1$, $\tau_2=-1$, $S_1(\tau_1)=\mathcal{E}_1(\tau_1)=\{\zeta_6,\zeta_6^2,-1,\zeta_6^4,\zeta_6^5,1\}$, and $S_1(\tau_2)=\mathcal{E}_1(\tau_2)=\{\zeta_{12},\sqrt{-1},\zeta_{12}^5,\zeta_{12}^7,-\sqrt{-1},\zeta_{12}^{11}\}$. Then $\mathcal{F}_1(\tau_1)=\{\zeta_6,\zeta_6^2,-1,\zeta_6^4,\zeta_6^5\}$ and $\mathcal{F}_1(\tau_2)=\{\zeta_{12},\sqrt{-1},\zeta_{12}^5,\zeta_{12}^7,-\sqrt{-1},\zeta_{12}^{11},-1\}$.

We show the equality $s_1s_2s_3s_4^{-1}s_5^{-1}=\sqrt{5}/2$ by proving a general relation among the functions $\Phi_i(z)$ defined by (3.2). Let $d\geq 6$ be an even integer. Putting

$$p_0=-2,\ p_1=-(\zeta_d+\overline{\zeta_d}),\ p_2=-(\zeta_d^2+\overline{\zeta_d}^2),\ldots,\ p_{\frac{d}{2}}=-(\zeta_d^{\frac{d}{2}}+\overline{\zeta_d}^{\frac{d}{2}})=2$$

in the equation (3.2) with $b=1$, we have

$$\begin{aligned}&\left(\Phi_0(z)\cdot\Phi_1^2(z)\Phi_2^2(z)\cdots\Phi_{\frac{d}{2}-1}^2(z)\cdot\Phi_{\frac{d}{2}}(z)\right)\cdot\Phi_0^{-1}(z)\\ &=\prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}}\frac{(z^{d^{k+1}}-1)^2}{(z^{d^k}-1)^2}\right)=\frac{1}{(z-1)^2}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2d^k})^{d-1}}.\end{aligned}$$

In the same way, putting

$$p_{\frac{d}{2}+1}=-(\zeta_{2d}+\overline{\zeta_{2d}}),\ p_{\frac{d}{2}+2}=-(\zeta_{2d}^3+\overline{\zeta_{2d}}^3),\ldots,\ p_d=-(\zeta_{2d}^{d-1}+\overline{\zeta_{2d}}^{d-1}),$$

we get

$$\begin{aligned}&\Phi_{\frac{d}{2}+1}^2(z)\Phi_{\frac{d}{2}+2}^2(z)\cdots\Phi_d^2(z)\cdot\Phi_{\frac{d}{2}}^{-1}(z)\\ &=\prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}}\frac{(z^{d^{k+1}}+1)^2}{(z^{d^k}+1)^2}\right)=\frac{1}{(z+1)^2}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2d^k})^{d-1}}.\end{aligned}$$

Hence, noting that $\Phi_i(0) = 1$ $(1 \le i \le d)$, we have

$$\Phi(z) := \frac{\Phi_1(z)\Phi_2(z)\cdots\Phi_{\frac{d}{2}}(z)}{\Phi_{\frac{d}{2}+1}(z)\Phi_{\frac{d}{2}+2}(z)\cdots\Phi_d(z)} = \frac{1+z}{1-z}. \tag{3.7}$$

Now assume that $d = 6$ in (3.7). Noting that $p_5 = 0$ and putting

$$a_1 = p_1 = -1, \quad a_2 = p_2 = 1, \quad a_3 = p_3 = 2, \quad a_4 = p_4 = -\sqrt{3}, \quad a_5 = p_6 = \sqrt{3}$$

in the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$, we have

$$\Phi(\alpha^{-6}) = \frac{s_1 s_2 s_3}{s_4 s_5} = \frac{\alpha^6+1}{\alpha^6-1}.$$

Since $\alpha^6 = 8\alpha + 5$, we get

$$\frac{s_1 s_2 s_3}{s_4 s_5} = \frac{\alpha^6+1}{\alpha^6-1} = \frac{1}{2}(2\alpha - 1) = \frac{\sqrt{5}}{2}.$$

The transcendence degree is obtained in the same way as in the proof of Example 2.5. $\square$

Proof of Example 2.7. We use the property 3 of Theorem 1.2. Let $\tau_1 = \sqrt{-1}$, $\tau_2 = 1$,

$$S_2(\tau_1) = \{\zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61}\},$$

and

$$S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}.$$

Then

$$\Lambda_1(\tau_1) = \{\zeta_{16}^3, \zeta_{16}^{13}\},\ \Gamma_1(\tau_1) = \{\zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15}\},\ \Lambda_0(\tau_1) = \{\sqrt{-1}, -\sqrt{-1}\},$$

and so

$$\mathcal{F}_2(\tau_1) = \{\zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61}, \zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15}, \sqrt{-1}, -\sqrt{-1}\},$$
$$\mathcal{F}_1(\tau_2) = \{-1, \sqrt{-1}, -\sqrt{-1}\}.$$

Putting

$$p_1 = -(\zeta_{16}^1 + \zeta_{16}^{15}), \quad p_2 = -(\zeta_{16}^5 + \zeta_{16}^{11}), \quad p_3 = -(\zeta_{16}^7 + \zeta_{16}^9),$$

$$p_4 = -(\zeta_{64}^3 + \zeta_{64}^{61}), \quad p_5 = -(\zeta_{64}^{13} + \zeta_{64}^{51}), \quad p_6 = -(\zeta_{64}^{19} + \zeta_{64}^{45}), \quad p_7 = -(\zeta_{64}^{29} + \zeta_{64}^{35})$$

in the equation (3.2) with $b = 1$, we get

$$\begin{aligned}&\Phi_1(z)\Phi_2(z)\cdots\Phi_7(z)\\ &= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2\cdot 4^k})^6}\frac{(z^{4^{k+1}} - \zeta_{16}^3)(z^{4^{k+1}} - \zeta_{16}^{13})}{(z^{4^k} - \zeta_{16}^3)(z^{4^k} - \zeta_{16}^{13})}\frac{z^{2\cdot 4^{k+1}}+1}{z^{2\cdot 4^k}+1}\right)\end{aligned}$$

$$= \frac{1}{(z^2+1)(z-\zeta_{16}^3)(z-\zeta_{16}^{13})} \prod_{k=0}^{\infty} \frac{1}{(1+z^{2\cdot 4^k})^6}.$$

Letting $p_8 = 2$ and using (3.5) in the proof of Example 2.5, we have

$$\Phi(z) := \frac{\Phi_1(z)\Phi_2(z)\cdots\Phi_7(z)}{\Phi_8^2(z)} = \frac{(z-1)^4}{(z^2+1)(z-\zeta_{16}^3)(z-\zeta_{16}^{13})}.$$

Putting $a_i = p_i$ $(1 \le i \le 8)$ in the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$ and using (3.6), we obtain

$$\begin{aligned}
\frac{s_1\cdots s_7}{s_8^2} &= \Phi(\alpha^{-4}) = \frac{(\alpha^4-1)^4}{(\alpha^8+1)(\alpha^8+1-(\zeta_{16}^3+\zeta_{16}^{13})\alpha^4)} \\
&= \frac{(7\alpha^4-2\alpha^4)^2}{7\alpha^4(7\alpha^4-(\zeta_{16}^3+\zeta_{16}^{13})\alpha^4)} \\
&= \frac{25}{7(7-\sqrt{2-\sqrt{2}})},
\end{aligned}$$

since $\zeta_{16}^3 + \zeta_{16}^{13} = 2\cos(3\pi/8) = \sqrt{2-\sqrt{2}}$. □

Proof of Example 2.8. Let $d \ge 2$ be an integer. Let γ and η be complex numbers with $|\gamma| = |\eta| = 1$. We show a general relation which holds for the functions $\Phi_i(z)$ $(1 \le i \le 2d+2)$ defined by (3.2). Putting

$$p_1 = -(\gamma+\overline{\gamma}),\ p_2 = -(\gamma\zeta_d + \overline{\gamma\zeta_d}), \ldots,\ p_d = -(\gamma\zeta_d^{d-1} + \overline{\gamma\zeta_d^{d-1}}),$$

and $p_{d+1} = -(\gamma^d + \overline{\gamma}^d)$ in the equation (3.2) with $b = 1$, we have

$$\begin{aligned}
&\Phi_1(z)\cdots\Phi_d(z)\Phi_{d+1}^{-1}(z) \\
&= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}} \frac{1}{\left(1+p_{d+1}z^{d^k}+z^{2d^k}\right)} \prod_{i=1}^{d}(1+p_i z^{d^k}+z^{2d^k})\right) \\
&= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}(z^{d^k}-\gamma^d)(z^{d^k}-\overline{\gamma}^d)} \prod_{i=0}^{d-1}(z^{d^k}-\gamma\zeta_d^i)(z^{d^k}-\overline{\gamma\zeta_d^i})\right) \\
&= \frac{1}{(z-\gamma^d)(z-\overline{\gamma}^d)} \prod_{k=0}^{\infty}\frac{1}{(1+z^{2d^k})^{d-1}}.
\end{aligned}$$

Moreover, putting

$$p_{d+2} = -(\eta+\overline{\eta}),\ p_{d+3} = -(\eta\zeta_d + \overline{\eta\zeta_d}), \ldots,\ p_{2d+1} = -(\eta\zeta_d^{d-1} + \overline{\eta\zeta_d^{d-1}}),$$

and $p_{2d+2} = -(\eta^d + \overline{\eta}^d)$, we get

$$\Phi(z) := \frac{\Phi_1(z)\cdots\Phi_d(z)}{\Phi_{d+1}(z)} \cdot \frac{\Phi_{2d+2}(z)}{\Phi_{d+2}(z)\cdots\Phi_{2d+1}(z)} = \frac{(z-\eta^d)(z-\overline{\eta}^d)}{(z-\gamma^d)(z-\overline{\gamma}^d)}. \tag{3.8}$$

Substituting $z = \alpha^{-4}$ into (3.8) and using (3.6), we get

$$\Phi(\alpha^{-4}) = \frac{\alpha^8 + p_{2d+2}\alpha^4 + 1}{\alpha^8 + p_{d+1}\alpha^4 + 1} = \frac{7 + p_{2d+2}}{7 + p_{d+1}}. \tag{3.9}$$

For Example 2.8, we take $d = 4$ and

$$\gamma = \frac{3 + \sqrt{-7}}{4}, \qquad \eta = \frac{2 + \sqrt{-1}}{\sqrt{5}}.$$

Noting that $\gamma^4 \neq \eta^4$ and taking $\tau_1 = \gamma^4$ and $\tau_2 = \eta^4$ in the property 3 of Theorem 1.2, we have

$$\begin{aligned} S_1(\tau_1) = \mathcal{E}_1(\tau_1) &= \{\gamma, \sqrt{-1}\gamma, -\gamma, -\sqrt{-1}\gamma, \overline{\gamma}, \sqrt{-1}\overline{\gamma}, -\overline{\gamma}, -\sqrt{-1}\overline{\gamma}\}, \\ S_1(\tau_2) = \mathcal{E}_1(\tau_2) &= \{\eta, \sqrt{-1}\eta, -\eta, -\sqrt{-1}\eta, \overline{\eta}, \sqrt{-1}\overline{\eta}, -\overline{\eta}, -\sqrt{-1}\overline{\eta}\}, \end{aligned}$$

and so

$$\begin{aligned} \mathcal{F}_1(\tau_1) &= \{\gamma, \sqrt{-1}\gamma, -\gamma, -\sqrt{-1}\gamma, \overline{\gamma}, \sqrt{-1}\overline{\gamma}, -\overline{\gamma}, -\sqrt{-1}\overline{\gamma}, \gamma^4, \overline{\gamma}^4\}, \\ \mathcal{F}_1(\tau_2) &= \{\eta, \sqrt{-1}\eta, -\eta, -\sqrt{-1}\eta, \overline{\eta}, \sqrt{-1}\overline{\eta}, -\overline{\eta}, -\sqrt{-1}\overline{\eta}, \eta^4, \overline{\eta}^4\}, \end{aligned}$$

since γ and η are not roots of unity. Then we have

$$p_1 = -\frac{3}{2},\ p_2 = \frac{\sqrt{7}}{2},\ p_3 = \frac{3}{2},\ p_4 = -\frac{\sqrt{7}}{2},\ p_5 = \frac{31}{16},$$
$$p_6 = -\frac{4}{\sqrt{5}},\ p_7 = \frac{2}{\sqrt{5}},\ p_8 = \frac{4}{\sqrt{5}},\ p_9 = -\frac{2}{\sqrt{5}},\ p_{10} = \frac{14}{25},$$

since

$$\gamma^4 = -\frac{31 - 3\sqrt{-7}}{32}, \qquad \eta^4 = -\frac{7 - 24\sqrt{-1}}{25}.$$

Using (3.9), we get

$$\frac{s_1 \cdots s_4}{s_5} \cdot \frac{s_{10}}{s_6 \cdots s_9} = \frac{7 + 14/25}{7 + 31/16} = \frac{3024}{3575}$$

by the equation (3.3) with $N = 1$. The transcendence degree is obtained in the same way as in the proof of Example 2.5. □

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