

Polynomials whose coefficients are k -Fibonacci numbers

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Abstract

Let $\{a_n\}_{n \geq 0}$ denote the linear recursive sequence of order k ($k \geq 2$) defined by the initial values $a_0 = a_1 = \dots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the recursion $a_n = a_{n-1} + a_{n-2} + \dots + a_{n-k}$ if $n \geq k$. The a_n are often called k -Fibonacci numbers and reduce to the usual Fibonacci numbers when $k = 2$. Let $P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \dots + a_{n+k-2}x + a_{n+k-1}$, which we will refer to as a k -Fibonacci coefficient polynomial. In this paper, we show for all k that the polynomial $P_{n,k}(x)$ has no real zeros if n is even and exactly one real zero if n is odd. This generalizes the known result for the $k = 2$ and $k = 3$ cases corresponding to Fibonacci and Tribonacci coefficient polynomials, respectively. It also improves upon a previous upper bound of approximately k for the number of real zeros of $P_{n,k}(x)$. Finally, we show for all k that the sequence of real zeros of the polynomials $P_{n,k}(x)$ when n is odd converges to the opposite of the positive zero of the characteristic polynomial associated with the sequence a_n . This generalizes a previous result for the case $k = 2$.

Keywords: k -Fibonacci sequence, zeros of polynomials, linear recurrences

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1. Introduction

Let the recursive sequence $\{a_n\}_{n \geq 0}$ of order k ($k \geq 2$) be defined by the initial values $a_0 = a_1 = \dots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the linear recursion

$$a_n = a_{n-1} + a_{n-2} + \dots + a_{n-k}, \quad n \geq k. \quad (1.1)$$

The numbers a_n are sometimes referred to as k -Fibonacci numbers (or *generalized Fibonacci* numbers) and reduce to the usual *Fibonacci* numbers F_n when $k = 2$ and to the *Tribonacci* numbers T_n when $k = 3$. (See, e.g., A000045 and A000073 in [11].) The sequence a_n was first considered by Knuth [3] and has been a topic of study in enumerative combinatorics. See, for example, [1, Chapter 3] or [9] for interpretations of a_n in terms of linear tilings or k -filtering linear partitions, respectively, and see [10] for a q -generalization of a_n .

Garth, Mills, and Mitchell [2] introduced the definition of the Fibonacci coefficient polynomials $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and—among other things—determined the number of real zeros of $p_n(x)$. In particular, they showed that $p_n(x)$ has no real zeros if n is even and exactly one real zero if n is odd. Later, this result was extended by Mátyás [5, 6] to more general second order recurrences. The same result also holds for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$, which was shown by Mátyás and Szalay [8].

If $k \geq 2$ and $n \geq 1$, then define the polynomial $P_{n,k}(x)$ by

$$P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}. \quad (1.2)$$

We will refer to $P_{n,k}(x)$ as a k -Fibonacci coefficient polynomial. Note that when $k = 2$ and $k = 3$, the $P_{n,k}(x)$ reduce to the Fibonacci and Tribonacci coefficient polynomials $p_n(x)$ and $q_n(x)$ mentioned above. In [7], the following result was obtained concerning the number of real zeros of $P_{n,k}(x)$ as a corollary to a more general result involving sequences defined by linear recurrences with non-negative integral weights.

Theorem 1.1. *Let h denote the number of real zeros of the polynomial $P_{n,k}(x)$ defined by (1.2) above. Then we have*

- (i) $h = k - 2 - 2j$ for some $j = 0, 1, \dots, (k - 2)/2$, if k and n are even,
- (ii) $h = k - 1 - 2j$ for some $j = 0, 1, \dots, (k - 2)/2$, if k is even and n is odd,
- (iii) $h = k - 1 - 2j$ for some $j = 0, 1, \dots, (k - 1)/2$, if k is odd and n is even,
- (iv) $h = k - 2j$ for some $j = 0, 1, \dots, (k - 1)/2$, if k and n are odd.

For example, Theorem 1.1 states when $k = 3$ that the number of real zeros of the polynomial $P_{n,3}(x)$ is either 0 or 2 if n is even or 1 or 3 if n is odd. As already mentioned, it was shown in [8] that $P_{n,3}(x)$ possesses no real zeros when n is even and exactly one real zero when n is odd.

In this paper, we show that the polynomial $P_{n,k}(x)$ possesses the smallest possible number of real zeros in every case and prove the following result.

Theorem 1.2. *Let $k \geq 2$ be a positive integer and $P_{n,k}(x)$ be defined by (1.2) above. Then we have the following:*

- (i) *If n is even, then $P_{n,k}(x)$ has no real zeros.*
- (ii) *If n is odd, then $P_{n,k}(x)$ has exactly one real zero.*

We prove Theorem 1.2 as a series of lemmas in the third and fourth sections below, and have considered separately the cases for even and odd k . Combining

Theorems 3.5 and 4.5 below gives Theorem 1.2. The crucial steps in our proofs of Theorems 3.5 and 4.5 are Lemmas 3.2 and 4.2, respectively, where we make a comparison of consecutive derivatives of a polynomial evaluated at the point $x = 1$. This allows us to show that there is exactly one zero when $x \leq -1$ in the case when n is odd. We remark that our proof, when specialized to the cases $k = 2$ and $k = 3$, provides an alternative proof to the ones given in [2] and [8], respectively, in these cases. In the final section, we show for all k that the sequence of real zeros of the polynomials $P_{n,k}(x)$ for n odd converges to $-\lambda$, where λ is the positive zero of the characteristic polynomial associated with the sequence a_n (see Theorem 5.5 below). This generalizes the result for the $k = 2$ case, which was shown in [2].

2. Preliminaries

We seek to determine the number of real zeros of the polynomial $P_{n,k}(x)$. By the following lemma, we may restrict our attention to the case when $x \leq -1$.

Lemma 2.1. *If $k \geq 2$ and $n \geq 1$, then the polynomial $P_{n,k}(x)$ has no zeros on the interval $(-1, \infty)$.*

Proof. Clearly, the equation $P_{n,k}(x) = 0$ has no roots if $x \geq 0$ since it has positive coefficients. Suppose $-1 < x < 0$. If n is odd, then

$$a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1} > 0, \quad 0 \leq j \leq (n-1)/2,$$

since $x^{n-2j-1} > -x^{n-2j} > 0$ if $-1 < x < 0$ and $a_{k+2j} \geq a_{k+2j-1} > 0$. This implies

$$P_{n,k}(x) = \sum_{j=0}^{\frac{n-1}{2}} (a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1}) > 0.$$

Similarly, if n is even, then

$$P_{n,k}(x) = a_{k-1}x^n + \sum_{j=0}^{\frac{n-2}{2}} (a_{k+2j}x^{n-2j-1} + a_{k+2j+1}x^{n-2j-2}) > 0.$$

□

So we seek the zeros of $P_{n,k}(x)$ where $x \leq -1$, equivalently, the zeros of $P_{n,k}(-x)$ where $x \geq 1$. For this, it is more convenient to consider the zeros of $g_{n,k}(x)$ given by

$$g_{n,k}(x) := c_k(-x)P_{n,k}(-x), \tag{2.1}$$

see [7], where

$$c_k(x) := x^k - x^{k-1} - x^{k-2} - \dots - x - 1 \tag{2.2}$$

denotes the *characteristic polynomial* associated with the sequence a_n .

By [7, Lemma 2.1], we have

$$\begin{aligned} g_{n,k}(x) &= (-x)^{n+k} - a_{n+k}(-x)^{k-1} - (a_{n+1} + a_{n+2} + \cdots + a_{n+k-1})(-x)^{k-2} \\ &\quad - \cdots - (a_{n+k-2} + a_{n+k-1})(-x) - a_{n+k-1} \\ &= (-x)^{n+k} - a_{n+k}(-x)^{k-1} - \sum_{r=1}^{k-1} \left(\sum_{j=r}^{k-1} a_{n+j} \right) (-x)^{k-r-1}. \end{aligned} \quad (2.3)$$

We now wish to study the zeros of $g_{n,k}(x)$, where $x \geq 1$. In the subsequent two sections, we undertake such a study, considering separately the even and odd cases for k .

3. The case k even

Throughout this section, k will denote a positive even integer. We consider the zeros of the polynomial $g_{n,k}(x)$ where $x \geq 1$, and for this, it is more convenient to consider the zeros of the polynomial

$$f_{n,k}(x) := (1+x)g_{n,k}(x), \quad (3.1)$$

where $x \geq 1$.

First suppose n is odd. Note that when k is even and n is odd, we have

$$\begin{aligned} f_{n,k}(x) &= -x^{n+k}(1+x) + a_{n+k}x^k + a_n x^{k-1} - a_{n+1}x^{k-2} + a_{n+2}x^{k-3} \\ &\quad - \cdots + a_{n+k-2}x - a_{n+k-1} \\ &= -x^{n+k}(1+x) + a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r} x^{k-r-1}, \end{aligned} \quad (3.2)$$

by (2.3) and the recurrence for a_n . In the lemmas below, we ascertain the number of the zeros of the polynomial $f_{n,k}(x)$ when $x \geq 1$. We will need the following combinatorial inequality.

Lemma 3.1. *If $k \geq 4$ is even and $n \geq 1$, then*

$$a_{n+k+1} \geq \sum_{r=0}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1}. \quad (3.3)$$

Proof. We have

$$\begin{aligned} a_{n+k+1} &= a_{n+k} + \sum_{r=1}^{k-1} a_{n+r} \geq 2 \sum_{r=1}^{k-1} a_{n+r} \\ &= 2a_{n+k-1} + 2a_{n+k-2} + 2 \sum_{r=1}^{k-3} a_{n+r} \geq 2a_{n+k-1} + 4 \sum_{r=1}^{k-3} a_{n+r} \end{aligned}$$

$$\begin{aligned}
 &= 2a_{n+k-1} + 4a_{n+k-3} + 4a_{n+k-4} + 4 \sum_{r=1}^{k-5} a_{n+r} \\
 &\geq 2a_{n+k-1} + 4a_{n+k-3} + 8 \sum_{r=1}^{k-5} a_{n+r} \\
 &= \dots \geq \sum_{r=i}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+1} \sum_{r=1}^{2i-1} a_{n+r} \\
 &= \sum_{r=i-1}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+1} a_{n+2i-2} + 2^{\frac{k}{2}-i+1} \sum_{r=1}^{2i-3} a_{n+r} \\
 &\geq \sum_{r=i-1}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1} + 2^{\frac{k}{2}-i+2} \sum_{r=1}^{2i-3} a_{n+r} \\
 &= \dots \geq \sum_{r=0}^{\frac{k}{2}-1} 2^{\frac{k}{2}-r} a_{n+2r+1},
 \end{aligned}$$

which gives (3.3). □

The following lemma will allow us to determine the number of zeros of $f_{n,k}(x)$ for $x \geq 1$.

Lemma 3.2. *Suppose $k \geq 4$ is even and n is odd. If $1 \leq i \leq k-1$, then $f_{n,k}^{(i)}(1) < 0$ implies $f_{n,k}^{(i+1)}(1) < 0$, where $f_{n,k}^{(i)}$ denotes the i -th derivative of $f_{n,k}$.*

Proof. Let $f = f_{n,k}$ and $i = k - j$ for some $1 \leq j \leq k - 1$. Then the assumption $f^{(k-j)}(1) < 0$ is equivalent to

$$\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} < \prod_{s=1}^{k-j} (n+j+s) + \prod_{s=1}^{k-j} (n+j+s+1). \quad (3.4)$$

We will show that inequality (3.4) implies

$$\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} < \prod_{s=0}^{k-j} (n+j+s) + \prod_{s=0}^{k-j} (n+j+s+1). \quad (3.5)$$

Observe first that the left-hand side of both inequalities (3.4) and (3.5) is positive as

$$\begin{aligned}
 &\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \\
 &= \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} > 0,
 \end{aligned}$$

since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $\frac{k!}{j!} > \frac{(k-r-1)!}{(j-r-1)!}$. Note also that

$$\frac{\prod_{s=0}^{k-j} (n+j+s) + \prod_{s=0}^{k-j} (n+j+s+1)}{\prod_{s=1}^{k-j} (n+j+s) + \prod_{s=1}^{k-j} (n+j+s+1)} > n+j,$$

so to show (3.5), it suffices to show

$$\begin{aligned} \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\ \leq (n+j) \left(\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right). \end{aligned} \quad (3.6)$$

For (3.6), it is enough to show

$$\begin{aligned} \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\ \leq (j+1) \left(\frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right). \end{aligned} \quad (3.7)$$

Starting with the left-hand side of (3.7), we have

$$\begin{aligned} \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\ = \frac{k!}{(j-1)!} \sum_{r=j-1}^{k-1} a_{n+r} + \sum_{r=0}^{j-2} \left(\frac{k!}{(j-1)!} + (-1)^r \frac{(k-r-1)!}{(j-r-2)!} \right) a_{n+r} \\ = \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left(j \frac{k!}{j!} + (-1)^r (j-r-1) \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\ = \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\ + \sum_{r=0}^{j-1} (-1)^{r+1} (r+1) \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \\ \leq \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\ + \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r+2) \frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1} \end{aligned}$$

$$\begin{aligned}
 &= (j+1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
 &\quad - \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r+2) \frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1}.
 \end{aligned}$$

Below we show

$$\sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r+2) \frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1} \leq \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r}. \quad (3.8)$$

Then from (3.8), we have

$$\begin{aligned}
 &\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
 &\leq (j+1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
 &\leq (j+1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} (j+1) \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
 &= (j+1) \frac{k!}{j!} a_{n+k} + (j+1) \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r},
 \end{aligned}$$

which gives (3.7), as desired.

To finish the proof, we need to show (3.8). We may assume $j \geq 2$, since the inequality is trivial when $j = 1$. By Lemma 3.1 and the fact that $2^m \geq 2m$ if $m \geq 1$, we have

$$\begin{aligned}
 &\sum_{r=j}^{k-1} a_{n+r} \geq a_{n+k-1} \\
 &\geq \sum_{r=0}^{\frac{k}{2}-2} 2^{\frac{k}{2}-r-1} a_{n+2r+1} \geq \sum_{r=0}^{\frac{k}{2}-2} (k-2r-2) a_{n+2r+1} \geq \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (k-2r-2) a_{n+2r+1},
 \end{aligned}$$

the last inequality holding since $j \leq k-1$, with k even. So to show (3.8), it is enough to show

$$(k-2r-2) \frac{k!}{j!} \geq (2r+2) \frac{(k-2r-2)!}{(j-2r-2)!}, \quad 0 \leq r \leq \lfloor (j-2)/2 \rfloor, \quad (3.9)$$

where $2 \leq j \leq k-1$. Since the ratio $\frac{k!/j!}{(k-2r-2)!/(j-2r-2)!}$ is decreasing in j for fixed k and r , one needs to verify (3.9) only when $j = k-1$, and it holds in this case since $2r+2 \leq j < k$. This completes the proof. \square

We now determine the number of zeros of $f_{n,k}(x)$ on the interval $[1, \infty)$.

Lemma 3.3. *Suppose $k \geq 4$ is even and n is odd. Then the polynomial $f_{n,k}(x)$ has exactly one zero on the interval $[1, \infty)$. Furthermore, this zero is simple.*

Proof. Let $f = f_{n,k}$, where we first assume $n \geq 3$. Then

$$f(1) = -2 + a_{n+k} + \sum_{r=0}^{k-1} (-1)^r a_{n+r} = -2 + 2 \sum_{r=0}^{\frac{k}{2}-1} a_{n+2r} > 0,$$

since $a_{n+k-2} \geq a_{k+1} = 2$. Let ℓ be the smallest positive integer i such that $f^{(i)}(1) < 0$; note that $1 \leq \ell \leq k+1$ since $f^{(k+1)}(1) < 0$. Then

$$f^{(\ell+1)}(1), f^{(\ell+2)}(1), \dots, f^{(k+1)}(1)$$

are all negative, by Lemma 3.2. Since $f^{(k+1)}(x) < 0$ for all $x \geq 1$, it follows that $f^{(\ell)}(x) < 0$ for all $x \geq 1$. To see this, note that if $\ell \leq k$, then $f^{(k)}(1) < 0$ implies $f^{(k)}(x) < 0$ for all $x \geq 1$, which in turn implies each of $f^{(k)}(x), f^{(k-1)}(x), \dots, f^{(\ell)}(x)$ is negative for all $x \geq 1$.

If $\ell \geq 2$, then $f^{(\ell-1)}(1) \geq 0$ and $f^{(\ell)}(x) < 0$ for all $x \geq 1$. Since $f^{(\ell-1)}(1) \geq 0$ and $\lim_{x \rightarrow \infty} f^{(\ell-1)}(x) = -\infty$, we have either (i) $f^{(\ell-1)}(1) = 0$ and $f^{(\ell-1)}(x)$ has no zeros on the interval $(1, \infty)$ or (ii) $f^{(\ell-1)}(1) > 0$ and $f^{(\ell-1)}(x)$ has exactly one zero on the interval $(1, \infty)$. If $\ell \geq 3$, then $f^{(\ell-2)}(x)$ would also have at most one zero on $(1, \infty)$ since $f^{(\ell-2)}(1) \geq 0$, with $f^{(\ell-2)}(x)$ initially increasing up to some point $s \geq 1$ before it decreases monotonically to $-\infty$ (where $s = 1$ if $f^{(\ell-1)}(1) = 0$ and $s > 1$ if $f^{(\ell-1)}(1) > 0$). Note that each derivative of f of order ℓ or less is eventually negative. Continuing in this fashion, we then see that if $\ell \geq 2$, then $f'(x)$ has at most one zero on the interval $(1, \infty)$, with $f'(1) \geq 0$ and $f'(x)$ eventually negative. If $\ell = 1$, then $f'(x) < 0$ for all $x \geq 1$. Since $f(1) > 0$ and $\lim_{x \rightarrow \infty} f(x) = -\infty$, it follows in either case that f has exactly one zero on the interval $[1, \infty)$, which finishes the case when $n \geq 3$.

If $n = 1$, then $f_{1,k}(x) = -x^{k+1}(1+x) + 2x^k + x - 1$ so that $f_{1,k}(1) = 0$, with

$$\begin{aligned} f'_{1,k}(x) &= -(k+1)x^k - (k+2)x^{k+1} + 2kx^{k-1} + 1 \\ &\leq -(k+1)x^{k-1} - (k+2)x^{k-1} + 2kx^{k-1} + 1 = -3x^{k-1} + 1 < 0 \end{aligned}$$

for $x \geq 1$. Thus, there is exactly one zero on the interval $[1, \infty)$ in this case as well.

Let t be the root of the equation $f_{n,k}(x) = 0$ on $[1, \infty)$. We now show that t has multiplicity one. First assume $n \geq 3$. Then $t > 1$. We consider cases depending on the value of $f'(1)$. If $f'(1) < 0$, then $f'(x) < 0$ for all $x \geq 1$ and thus $f'(t) < 0$ is non-zero, implying t is a simple root. If $f'(1) > 0$, then $f'(t) < 0$ due to $f(1) > 0$ and the fact that $f'(x)$ would then have one root v on $(1, \infty)$ with $v < t$. Finally, if $f'(1) = 0$, then the proof of Lemma 3.2 above shows that $f''(1) < 0$ and thus $f''(x) < 0$ for all $x \geq 1$, which implies $f'(t) < 0$. If $n = 1$, then $t = 1$ and $f'_{1,k}(1) < 0$. Thus, t is a simple root in all cases, as desired, which completes the proof. \square

We next consider the case when n is even.

Lemma 3.4. *Suppose $k \geq 4$ and n are even. Then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.*

Proof. In this case, we have

$$f_{n,k}(x) = x^{n+k}(1+x) + a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r}x^{k-r-1},$$

by (2.3) and (3.1). If $x \geq 1$, then $f_{n,k}(x) > 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $x^k \geq x^{k-r-1}$ for $0 \leq r \leq k-1$. \square

The main result of this section now follows rather quickly.

Theorem 3.5. *(i) If k is even and n is odd, then the polynomial $P_{n,k}(x)$ has one real zero q , and it is simple with $q \leq -1$.*

(ii) If k and n are even, then the polynomial $P_{n,k}(x)$ has no real zeros.

Proof. Note first that the preceding lemmas, where we assumed $k \geq 4$ is even, may be adjusted slightly and are also seen to hold in the case $k = 2$. First suppose n is odd. By Lemma 3.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero for $x \geq 1$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \dots - 1$ has one negative real zero when k is even, and it is seen to lie in the interval $(-1, 0)$. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(-x)$ has one zero for $x \geq 1$. Thus, $P_{n,k}(x)$ has one zero for $x \leq -1$, and it is simple. By Lemma 2.1, the polynomial $P_{n,k}(x)$ has exactly one real zero.

If n is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 3.4. By (2.1), it follows that $P_{n,k}(x)$ has no zeros for $x \leq -1$. By Lemma 2.1, $P_{n,k}(x)$ has no real zeros. \square

4. The case k odd

Throughout this section, $k \geq 3$ will denote a positive odd integer. We study the zeros of the polynomial $g_{n,k}(x)$ when $x \geq 1$, and for this, it is again more convenient to consider the polynomial $f_{n,k}(x) := (1+x)g_{n,k}(x)$. First suppose n is odd. When k and n are both odd, note that

$$\begin{aligned} f_{n,k}(x) &= x^{n+k}(1+x) - a_{n+k}x^k - a_n x^{k-1} + a_{n+1}x^{k-2} - \dots + a_{n+k-2}x - a_{n+k-1} \\ &= x^{n+k}(1+x) - a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^{r+1} a_{n+r}x^{k-r-1}, \end{aligned}$$

by (2.3) and the recurrence for a_n . In the lemmas below, we ascertain the number of zeros of the polynomial $f_{n,k}(x)$ when $x \geq 1$. We start with the following inequality.

Lemma 4.1. *Suppose $k \geq 3$ is odd and $n \geq 1$. If $1 \leq j \leq k-1$, then*

$$3 \frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} 2r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1}. \quad (4.1)$$

Proof. First note that we have the inequality

$$a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r} a_{n+2r}. \quad (4.2)$$

To show (4.2), proceed as in the proof of Lemma 3.1 above and write

$$\begin{aligned} a_{n+k-1} &\geq a_{n+k-2} + \sum_{r=2}^{k-3} a_{n+r} \\ &\geq 2a_{n+k-3} + 2 \sum_{r=2}^{k-4} a_{n+r} \\ &= 2a_{n+k-3} + 2a_{n+k-4} + 2 \sum_{r=2}^{k-5} a_{n+r} \\ &\geq 2a_{2n+k-3} + 4a_{n+k-5} + 4 \sum_{r=2}^{k-6} a_{n+r} \\ &= \dots \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r} a_{n+2r}. \end{aligned}$$

Since $2^m \geq 2m$ if $m \geq 1$, we have

$$a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2}-r} a_{n+2r} \geq \sum_{r=1}^{\frac{k-3}{2}} (k-2r-1) a_{n+2r}. \quad (4.3)$$

First suppose $j \leq k-2$. In this case, we show

$$\frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1}, \quad (4.4)$$

which implies (4.1). And (4.4) is seen to hold since by (4.3),

$$\frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} \frac{(k-2r-1)k!}{j!} a_{n+2r} \geq \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \frac{(k-2r-1)k!}{j!} a_{n+2r},$$

with $a_{n+2r} \geq a_{n+2r-1}$ and

$$\frac{(k-2r-1)k!}{r(k-2r)!} \geq \frac{(k-2)!}{(k-2r-2)!} \geq \frac{j!}{(j-2r)!}.$$

The $j = k - 1$ case of (4.1) follows from noting

$$\begin{aligned} 3ka_{n+k-1} &\geq ka_{n+k-1} + \sum_{r=1}^{\frac{k-3}{2}} 2k(k-2r-1)a_{n+2r} \\ &\geq (k-1)a_{n+k-2} + \sum_{r=1}^{\frac{k-3}{2}} 2r(k-2r)a_{n+2r-1} = \sum_{r=1}^{\frac{k-1}{2}} 2r(k-2r)a_{n+2r-1}, \end{aligned}$$

since $k(k-2r-1) \geq r(k-2r)$ if $1 \leq r \leq \frac{k-3}{2}$. \square

Lemma 4.2. *Suppose $k, n \geq 3$ are odd. If $1 \leq i \leq k-1$, then $f_{n,k}^{(i)}(1) > 0$ implies $f_{n,k}^{(i+1)}(1) > 0$.*

Proof. Let $f = f_{n,k}$ and $i = k - j$ for some $1 \leq j \leq k - 1$. Then the assumption $f^{(k-j)}(1) > 0$ is equivalent to

$$\frac{(n+k)!}{(n+j)!} + \frac{(n+k+1)!}{(n+j+1)!} > \frac{k!}{j!}a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r}. \quad (4.5)$$

Using (4.5), we will show $f^{(k-j+1)}(1) > 0$, i.e.,

$$\frac{(n+k)!}{(n+j-1)!} + \frac{(n+k+1)!}{(n+j)!} > \frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r}. \quad (4.6)$$

Note that the right-hand side of both inequalities (4.5) and (4.6) is positive since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$. Since the left-hand side of (4.6) divided by the left-hand side of (4.5) is greater than $n+j$, it suffices to show

$$\begin{aligned} &\frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\ &\leq (n+j) \left(\frac{k!}{j!}a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right). \end{aligned} \quad (4.7)$$

For (4.7), it is enough to show

$$\begin{aligned} &\frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\ &\leq (j+3) \left(\frac{k!}{j!}a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right), \end{aligned} \quad (4.8)$$

since $n \geq 3$.

Starting with the left-hand-side of (4.8), and proceeding at this stage as in the proof of Lemma 3.2 above, we have

$$\begin{aligned}
& \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
& \leq \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
& \quad + \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} 2r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1} \\
& = (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
& \quad - 3 \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} 2r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1} \\
& \leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r},
\end{aligned}$$

where the last inequality follows from Lemma 4.1. Thus,

$$\begin{aligned}
& \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
& \leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
& \leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} (j+3) \left(\frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
& = (j+3) \frac{k!}{j!} a_{n+k} + (j+3) \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r},
\end{aligned}$$

which gives (4.8) and completes the proof. \square

We can now determine the number of zeros of $f_{n,k}(x)$ on the interval $[1, \infty)$.

Lemma 4.3. *Suppose $k \geq 3$ and n are odd. Then $f_{n,k}(x)$ has exactly one zero on the interval $[1, \infty)$ and it is simple.*

Proof. If $n \geq 3$, then use Lemma 4.2 and the same reasoning as in the proof of Lemma 3.3 above. Note that in this case we have

$$f_{n,k}(1) = 2 - a_{n+k} + \sum_{r=0}^{k-1} (-1)^{r+1} a_{n+r} = 2 - 2 \sum_{r=0}^{\frac{k-1}{2}} a_{n+2r} < 0,$$

as $a_{n+k-1}, a_{n+k-3} > 0$. If $n = 1$, then $f_{1,k}(x) = x^{k+1}(1+x) - 2x^k + x - 1$ and the result also holds as $f_{1,k}(1) = 0$ with $f'_{1,k}(x) > 0$ if $x \geq 1$. \square

We next consider the case when n is even.

Lemma 4.4. *If $k \geq 3$ is odd and n is even, then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.*

Proof. In this case, we have

$$f_{n,k} = -x^{n+k}(1+x) - a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^{r+1} a_{n+r} x^{k-r-1}.$$

If $x \geq 1$, then $f_{n,k}(x) < 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $-x^k \leq -x^{k-r-1}$ for $0 \leq r \leq k-1$. \square

We now prove the main result of this section.

Theorem 4.5. (i) *If $k \geq 3$ and n are odd, then the polynomial $P_{n,k}(x)$ has one real zero q , and it is simple with $q \leq -1$.*

(ii) *If $k \geq 3$ is odd and n is even, then the polynomial $P_{n,k}(x)$ has no real zeros.*

Proof. First suppose n is odd. By Lemma 4.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero on $[1, \infty)$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \dots - 1$ has no negative real zeros when k is odd. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(x)$ has one zero for $x \leq -1$, and hence one real zero, by Lemma 2.1.

If n is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 4.4. Thus, neither does $P_{n,k}(-x)$, which implies it has no real zeros. \square

5. Convergence of zeros

In this section, we show that for each fixed $k \geq 2$, the sequence of real zeros of $P_{n,k}(x)$ for n odd is convergent. Before proving this, we remind the reader of the following version of Rouché's Theorem which can be found in [4].

Theorem 5.1 (Rouché). *If $p(z)$ and $q(z)$ are analytic interior to a simple closed Jordan curve \mathcal{C} , and are continuous on \mathcal{C} , with*

$$|p(z) - q(z)| < |q(z)|, \quad z \in \mathcal{C},$$

then the functions $p(z)$ and $q(z)$ have the same number of zeros interior to \mathcal{C} .

We now give three preliminary lemmas.

Lemma 5.2. (i) If $k \geq 2$, then the polynomial $c_k(x) = x^k - x^{k-1} - \dots - x - 1$ has one positive real zero λ , with $\lambda > 1$. All of its other zeros have modulus strictly less than one.

(ii) The zeros of $c_k(x)$, which we will denote by $\alpha_1 = \lambda, \alpha_2, \dots, \alpha_k$, are distinct and thus

$$a_n = c_1\alpha_1^n + c_2\alpha_2^n + \dots + c_n\alpha_k^n, \quad n \geq 0, \quad (5.1)$$

where c_1, c_2, \dots, c_k are constants.

(iii) The constant c_1 is a positive real number.

Proof. (i) It is more convenient to consider the polynomial $d_k(x) := (1-x)c_k(x)$. Note that

$$d_k(x) = (1-x) \left(x^k - \frac{1-x^k}{1-x} \right) = 2x^k - x^{k+1} - 1.$$

We regard $d_k(z)$ as a complex function. Since on the circle $|z| = 3$ in the complex plane holds

$$|2z^k| = 2 \cdot 3^k < 3^{k+1} - 1 = |-z^{k+1}| - 1 \leq |-z^{k+1} - 1|,$$

it follows from Rouché's Theorem that $d_k(z)$ has $k+1$ zeros in the disc $|z| < 3$ since the function $-z^{k+1} - 1$ has all of its zeros there. On the other hand, on the circle $|z| = 1 + \epsilon$, we have

$$|-z^{k+1}| = (1 + \epsilon)^{k+1} < 2(1 + \epsilon)^k - 1 \leq |2z^k - 1|,$$

which implies that the polynomial $d_k(z)$ has exactly k zeros in the disc $|z| < 1 + \epsilon$, for all $\epsilon > 0$ sufficiently small such that $-\frac{\ln(1-\epsilon)}{\ln(1+\epsilon)} < 2 \leq k$. Letting $\epsilon \rightarrow 0$, we see that there are k zeros for the polynomial $d_k(z)$ in the disc $|z| \leq 1$. But $z = 1$ is a zero of the polynomial $d_k(z) = (1-z)c_k(z)$ on the circle $|z| = 1$, and it is the only such zero since $d_k(z) = 0$ implies $|z|^k \cdot |2 - z| = 1$, or $|2 - z| = 1$, which is clearly satisfied by only $z = 1$. Hence, the polynomial $c_k(z)$ has $k-1$ zeros in the disc $|z| < 1$ and exactly one zero in the domain $1 < |z| < 3$. Finally, by Descartes' rule of signs and since $c_k(1) < 0$, we see that $c_k(x)$ has exactly one positive real zero λ , with $1 < \lambda < 3$.

(ii) We'll prove only the first statement, as the second one follows from the first and the theory of linear recurrences. For this, first note that $d'_k(x) = 0$ implies $x = 0, \frac{2k}{k+1}$. Now the only possible rational roots of the equation $d_k(x) = 0$ are ± 1 , by the rational root theorem. Thus $d_k\left(\frac{2k}{k+1}\right) = 0$ is impossible as $k \geq 2$, which implies $d_k(x)$ and $d'_k(x)$ cannot share a zero. Therefore, the zeros of $d_k(x)$, and hence of $c_k(x)$, are distinct.

(iii) Substitute $n = 0, 1, \dots, k-1$ into (5.1), and recall that $a_0 = a_1 = \dots = a_{k-2} = 0$ with $a_{k-1} = 1$, to obtain a system of linear equations in the variables c_1, c_2, \dots, c_k . Let A be the coefficient matrix for this system (where the equations are understood to have been written in the natural order) and let A' be the matrix obtained from A by replacing the first column of A with the vector $(0, \dots, 0, 1)$ of

length k . Now the transpose of A and of the $(k - 1) \times (k - 1)$ matrix obtained from A' by deleting the first column and the last row are seen to be Vandermonde matrices. Therefore, by Cramer's rule, we have

$$\begin{aligned} c_1 &= \frac{\det A'}{\det A} = \frac{(-1)^{k+1} \prod_{2 \leq i < j \leq k} (\alpha_j - \alpha_i)}{\prod_{1 \leq i < j \leq k} (\alpha_j - \alpha_i)} \\ &= \frac{1}{(-1)^{k-1} \prod_{j=2}^k (\alpha_j - \alpha_1)} = \frac{1}{\prod_{j=2}^k (\alpha_1 - \alpha_j)}. \end{aligned}$$

If $j \geq 2$, then either $\alpha_j < 0$ or α_j and α_ℓ are complex conjugates for some ℓ . Note that $\alpha_1 - \alpha_j > 0$ in the first case and

$$(\alpha_1 - \alpha_j)(\alpha_1 - \alpha_\ell) = (\alpha_1 - a)^2 + b^2 > 0$$

in the second, where $\alpha_j = a + bi$. Since all of the complex zeros of $c_k(x)$ which aren't real come in conjugate pairs, it follows that c_1 is a positive real number. \square

We give the zeros of $c_k(z)$ for $2 \leq k \leq 5$ as well as the value of the constant c_1 in Table 1 below, where \bar{z} denotes the complex conjugate of z .

k	The zeros of $c_k(z)$	The constant c_1
2	1.61803, -0.61803	0.44721
3	1.83928, $r_1 = -0.41964 + 0.60629i, \bar{r}_1$	0.18280
4	1.92756, $-0.77480, r_1 = -0.07637 + 0.81470i, \bar{r}_1$	0.07907
5	1.96594, $r_1 = 0.19537 + 0.84885i,$ $r_2 = -0.67835 + 0.45853i, \bar{r}_1, \bar{r}_2$	0.03601

Table 1: The zeros of $c_k(z)$ and the constant c_1 .

The next lemma concerns the location of the positive zero of the k -th derivative of $f_{n,k}(x)$.

Lemma 5.3. *Suppose $k \geq 2$ is fixed and n is odd. Let $s_n (= s_{n,k})$ be the zero of $f_{n,k}(x)$ on $[1, \infty)$, where $f_{n,k}(x)$ is given by (3.1), and let $t_n (= t_{n,k})$ be the positive zero of the k -th derivative of $f_{n,k}(x)$. Let λ be the positive zero of $c_k(x)$. Then we have*

- (i) $t_n < s_n$ for all odd n , and
- (ii) $t_n \rightarrow \lambda$ as n odd increases without bound.

Proof. Suppose k is even, the proof when k is odd being similar. Then $f_{n,k}$ is given by (3.2) above. Throughout the following proof, n will always represent an odd integer and $f = f_{n,k}$. Recall from Lemma 3.3 that f has exactly one zero on the interval $[1, \infty)$.

(i) By Descartes' rule of signs, the polynomial $f^{(k)}(x)$ has one positive real zero t_n . If $t_n < 1 \leq s_n$, then we are done, so let us assume $t_n \geq 1$. The condition $t_n \geq 1$,

or equivalently $f^{(k)}(1) \geq 0$, then implies $n \geq 3$, and thus $f(1) > 0$. (Indeed, $t_n \geq 1$ for all n sufficiently large since $a_{n+k} \sim c_1 \lambda^{n+k}$, with $\lambda > 1$.)

Now observe that $f^{(k)}(1) \geq 0$ implies $f^{(i)}(1) > 0$ for $1 \leq i \leq k-1$, as the proof of Lemma 3.2 above shows in fact that $f^{(i)}(1) \leq 0$ implies $f^{(i+1)}(1) < 0$. Since $f^{(i)}(1) > 0$ for $0 \leq i \leq k-1$ and $f^{(k)}(1) \geq 0$, it follows that each of the polynomials $f(x), f'(x), \dots, f^{(k)}(x)$ has exactly one zero on $[1, \infty)$ since $f^{(k+1)}(x) < 0$ for all $x \geq 1$. Furthermore, the zero of $f^{(i)}(x)$ on $[1, \infty)$ is strictly larger than the zero of $f^{(i+1)}(x)$ on $[1, \infty)$ for $0 \leq i \leq k-1$. In particular, the zero of $f(x)$ is strictly larger than the zero of $f^{(k)}(x)$, which establishes the first statement.

(ii) Let us assume n is large enough to ensure $t_n \geq 1$. Note that

$$\frac{f^{(k)}(x)}{k!} = -\binom{n+k}{k}x^n - \binom{n+k+1}{k}x^{n+1} + a_{n,k}$$

so that

$$-2\binom{n+k+1}{k}x^{n+1} + a_{n,k} \leq \frac{f^{(k)}(x)}{k!} \leq -2\binom{n+k}{k}x^n + a_{n,k}, \quad x \geq 1. \quad (5.2)$$

Setting $x = t_n$ in (5.2), and rearranging, then gives

$$\left(\frac{a_{n+k}}{2\binom{n+k+1}{k}}\right)^{1/(n+1)} \leq t_n \leq \left(\frac{a_{n+k}}{2\binom{n+k}{k}}\right)^{1/n}. \quad (5.3)$$

The second statement then follows from letting n tend to infinity in (5.3) and noting $\lim_{n \rightarrow \infty} (a_{n+k})^{1/n} = \lambda$ (as $a_{n+k} \sim c_1 \lambda^{n+k}$, by Lemma 5.2). \square

We will also need the following formula for an expression involving the zeros of $c_k(x)$.

Lemma 5.4. *If $\alpha_1 = \lambda, \alpha_2, \dots, \alpha_k$ are the zeros of $c_k(x)$, then*

$$\begin{aligned} & \sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} \mathcal{S}_j\{\alpha_2, \alpha_3, \dots, \alpha_k\} \\ &= \frac{k\lambda^{k+2} - (2k-1)\lambda^{k+1} - (k-1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1}{(\lambda-1)^2(\lambda+1)}, \end{aligned} \quad (5.4)$$

where $\mathcal{S}_j\{\alpha_2, \alpha_3, \dots, \alpha_k\}$ denotes the j -th symmetric function in the quantities $\alpha_2, \alpha_3, \dots, \alpha_k$ if $1 \leq j \leq k-1$, with $\mathcal{S}_0\{\alpha_2, \alpha_3, \dots, \alpha_k\} := 1$.

Proof. Let us assume k is even, the proof in the odd case being similar. First note that

$$(-1)^{i+1} = \mathcal{S}_i\{\alpha_1, \alpha_2, \dots, \alpha_k\} = \mathcal{S}_i\{\alpha_2, \dots, \alpha_k\} + \lambda \mathcal{S}_{i-1}\{\alpha_2, \dots, \alpha_k\}, \quad 1 \leq i \leq k,$$

which gives the recurrences

$$\mathcal{S}_{2r}\{\alpha_2, \dots, \alpha_k\} = -1 - \lambda \mathcal{S}_{2r-1}\{\alpha_2, \dots, \alpha_k\}, \quad 1 \leq r \leq (k-2)/2, \quad (5.5)$$

and

$$\mathcal{S}_{2r+1}\{\alpha_2, \dots, \alpha_k\} = 1 - \lambda \mathcal{S}_{2r}\{\alpha_2, \dots, \alpha_k\}, \quad 0 \leq r \leq (k-2)/2. \quad (5.6)$$

Iterating (5.5) and (5.6) yields

$$\begin{aligned} \mathcal{S}_{2r}\{\alpha_2, \dots, \alpha_k\} &= -(1 + \lambda + \dots + \lambda^{2r-1}) + \lambda^{2r} \\ &= -\frac{1 - 2\lambda^{2r} + \lambda^{2r+1}}{1 - \lambda}, \quad 1 \leq r \leq (k-2)/2, \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} \mathcal{S}_{2r+1}\{\alpha_2, \dots, \alpha_k\} &= (1 + \lambda + \dots + \lambda^{2r}) - \lambda^{2r+1} \\ &= \frac{1 - 2\lambda^{2r+1} + \lambda^{2r+2}}{1 - \lambda}, \quad 0 \leq r \leq (k-2)/2. \end{aligned} \quad (5.8)$$

Note that (5.7) also holds in the case when $r = 0$.

By (5.7) and (5.8), we then have

$$\begin{aligned} &\sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} \mathcal{S}_j\{\alpha_2, \alpha_3, \dots, \alpha_k\} \\ &= -\sum_{r=0}^{\frac{k}{2}-1} \lambda^{k-2r-1} \left(\frac{1 - 2\lambda^{2r} + \lambda^{2r+1}}{1 - \lambda} \right) - \sum_{r=0}^{\frac{k}{2}-1} \lambda^{k-2r-2} \left(\frac{1 - 2\lambda^{2r+1} + \lambda^{2r+2}}{1 - \lambda} \right) \\ &= \frac{1}{\lambda - 1} \sum_{r=0}^{\frac{k}{2}-1} (\lambda^{k-2r-1} - 2\lambda^{k-1} + \lambda^k) + \frac{1}{\lambda - 1} \sum_{r=0}^{\frac{k}{2}-1} (\lambda^{k-2r-2} - 2\lambda^{k-1} + \lambda^k) \\ &= \frac{\lambda}{\lambda - 1} \left(\frac{\lambda^k - 1}{\lambda^2 - 1} \right) + \frac{1}{\lambda - 1} \left(\frac{\lambda^k - 1}{\lambda^2 - 1} \right) - \frac{2k\lambda^{k-1}}{\lambda - 1} + \frac{k\lambda^k}{\lambda - 1}, \end{aligned}$$

which gives (5.4). □

We now can prove the main result of this section.

Theorem 5.5. *Suppose $k \geq 2$ and n is odd. Let $r_n (= r_{n,k})$ denote the real zero of the polynomial $P_{n,k}(x)$ defined by (1.2) above. Then $r_n \rightarrow -\lambda$ as $n \rightarrow \infty$.*

Proof. Let n denote an odd integer throughout. We first consider the case when k is even. Equivalently, we show that $s_n \rightarrow \lambda$ as $n \rightarrow \infty$, where s_n denotes the zero of $f_{n,k}(x)$ on the interval $[1, \infty)$. By Lemma 5.3, we have $t_n < s_n$ for all n with $t_n \rightarrow \lambda$ as $n \rightarrow \infty$, where t_n is the positive zero of the k -th derivative of $f_{n,k}(x)$. So it is enough to show $s_n < \lambda$ for all n sufficiently large, i.e., $f_{n,k}(\lambda) < 0$.

By Lemma 5.2, we have

$$f_{n,k}(\lambda) = -\lambda^{n+k}(1 + \lambda) + a_{n,k}\lambda^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r}\lambda^{k-r-1}$$

$$\begin{aligned} &\sim -\lambda^{n+k}(1+\lambda) + c_1\lambda^{n+2k} + \sum_{r=0}^{k-1} (-1)^r c_1\lambda^{n+k-1} \\ &= \lambda^{n+k}(-1-\lambda+c_1\lambda^k), \end{aligned}$$

so that $f_{n,k}(\lambda) < 0$ for large n if $-1-\lambda+c_1\lambda^k < 0$, i.e.,

$$\lambda^k < \frac{1+\lambda}{c_1}. \quad (5.9)$$

So to complete the proof, we must show (5.9). By Lemmas 5.2 and 5.4, we have

$$\begin{aligned} \frac{1}{c_1} &= \prod_{j=2}^k (\lambda - \alpha_j) = \sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} \mathcal{S}_j\{\alpha_2, \alpha_3, \dots, \alpha_k\} \\ &= \frac{k\lambda^{k+2} - (2k-1)\lambda^{k+1} - (k-1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1}{(\lambda-1)^2(\lambda+1)}, \end{aligned}$$

so that (5.9) holds if and only

$$\lambda^k(\lambda-1)^2 < k\lambda^{k+2} - (2k-1)\lambda^{k+1} - (k-1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1,$$

i.e.,

$$1 + \lambda + k\lambda^k + (2k-3)\lambda^{k+1} < 2k\lambda^{k-1} + (k-1)\lambda^{k+2}. \quad (5.10)$$

Recall from the proof of Lemma 5.2 that $2\lambda^k = 1 + \lambda^{k+1}$. Substituting $\lambda^{k+1} = \frac{\lambda + \lambda^{k+2}}{2}$,

$$\lambda^k = \frac{1 + \frac{\lambda + \lambda^{k+2}}{2}}{2} = \frac{2 + \lambda + \lambda^{k+2}}{4},$$

and

$$\lambda^{k-1} = \frac{\lambda^k}{\lambda} = \frac{2 + \lambda + \lambda^{k+2}}{4\lambda}$$

into (5.10), and rearranging, then gives

$$\left(1 - \frac{\lambda}{2} - \frac{k}{\lambda}\right) + \frac{5k\lambda}{4} < \lambda^{k+2} \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2}\right). \quad (5.11)$$

For (5.11), note first that $c_k(2) > 0$ as $2^k > 2^k - 1 = 2^{k-1} + \dots + 1$, which implies $\lambda < 2 \leq k$ and thus $1 - \frac{\lambda}{2} - \frac{k}{\lambda} < 0$. So to show (5.11), it is enough to show

$$\frac{5k}{4} < \lambda^{k+1} \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2}\right). \quad (5.12)$$

For (5.12), we'll consider the cases $k=2$ and $k \geq 4$. If $k=2$, then $\lambda = \theta = \frac{1+\sqrt{5}}{2}$, so that (5.12) reduces in this case to $\frac{5}{2} < \theta^2 = \theta + 1$, which is true. Now suppose

$k \geq 4$ is even. First observe that $c_k\left(\frac{5}{3}\right) < 0$, whence $\lambda > \frac{5}{3}$, as $d_k\left(\frac{5}{3}\right) > 0$ since $\left(\frac{5}{3}\right)^k \left(2 - \frac{5}{3}\right) > 1$ for all $k \geq 3$. Thus, we have

$$\begin{aligned} \lambda^k &= (\lambda^{k-1} + 1) + \lambda^{k-2} + \lambda^{k-3} + \dots + \lambda \\ &> 2\lambda^{\frac{k-1}{2}} + \lambda^{k-2} + \lambda^{k-3} + \dots + \lambda > 2 \cdot \frac{5}{3} + \frac{5(k-2)}{3} = \frac{5k}{3}. \end{aligned}$$

So to show (5.12) when $k \geq 4$, it suffices to show

$$0 < \lambda \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2} \right) - \frac{3}{4} = \frac{k(2-\lambda)}{4} + \frac{2\lambda-3}{4},$$

which is true as $\frac{5}{3} < \lambda < 2$. This completes the proof in the even case.

If k is odd, then we proceed in a similar manner. Instead of inequality (5.9), we get

$$\lambda^k + \frac{1}{\lambda} < \frac{1+\lambda}{c_1}, \tag{5.13}$$

which is equivalent to

$$\left(1 - \frac{\lambda}{2} - \frac{k}{\lambda} + \frac{(\lambda-1)^2}{\lambda} \right) + \frac{5k\lambda}{4} < \lambda^{k+2} \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2} \right). \tag{5.14}$$

Note that the sum of the first four terms on the left-hand side of (5.14) is negative since $1 - \frac{k}{\lambda} < 0$ and $-\frac{\lambda}{2} + \frac{(\lambda-1)^2}{\lambda} < 0$ as $\frac{5}{3} < \lambda < 2$ for $k \geq 3$. Thus, it suffices to show (5.12) in the case when $k \geq 3$ is odd, which has already been done since the proof given above for it applies to all $k \geq 3$. \square

$n \setminus k$	2	3	4	5
1	1	1	1	1
5	1.39118	1.59674	1.61156	1.64627
9	1.48442	1.69002	1.73834	1.77122
49	1.59187	1.80885	1.88958	1.92625
99	1.60498	1.82403	1.90856	1.94605
199	1.61151	1.83165	1.91805	1.95599
λ	1.61803	1.83928	1.92756	1.96594

Table 2: Some real zeros of $P_{n,k}(-x)$, where λ is the positive zero of $c_k(x)$.

Perhaps the proofs presented here of Theorems 1.2 and 5.5 could be generalized to show comparable results for polynomials associated with linear recurrent sequences having various non-negative real weights, though the results are not true for all linear recurrences having such weights, as can be seen numerically in the case $k = 3$. Furthermore, numerical evidence (see Table 2 below) suggests that the sequence of zeros in Theorem 5.5 decreases monotonically for all k , as is true in the $k = 2$ case (see [2, Theorem 3.1]).

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